Applied Mathematics E-Notes, 11(2011), 255–260 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

On The Principal Equations Of Isospectral Beams*

Kazem Ghanbari†, Farshad Rezvan‡

Received 2 January 2011

Abstract

Applications of the jet space analysis to isospectral beams are considered. Using the jet space analysis, we show that the principal equations related to the beam equation have unique solutions.

1 Introduction

It is well known that there are different beams having the same spectrum under the clamped boundary conditions [2]. That is, if we consider the beam equation $u^{(4)} + (Au')' + Bu = \lambda u$ with clamped boundary conditions $u = u' = 0$, then we can find other classes of beam equations with different coefficients $\hat{A}$ and $\hat{B}$ with the same eigenvalues under the same clamped boundary conditions. For more details see [2].

Now, assume that we have two functions of t , say $\beta = \beta(t)$ and $\gamma = \gamma(t)$, where t stands for *time*. According to [2] we define two coefficients A and B as follows:

$$\begin{gathered} A = -\beta^2 - 2\beta\gamma - 2\gamma^2 - 3\beta' - 4\gamma', \\ B = \beta^2\gamma^2 + 2\beta\gamma^3 + \gamma^4 + \gamma^2\beta'^2\gamma' + 4\beta\gamma\gamma'4\gamma^2\gamma' + \beta'\gamma' + \gamma'^2 - \beta\beta'' - \gamma\beta'' - \beta''' - \gamma'''. \end{gathered} \tag{1}$$

The related beam principal equation can be stated as the following: Find r and s as functions of β and γ and their derivatives which satisfies these equations:

$$\begin{gathered} 2s + r'^2 = A, \\ s^2 + s'' - r's - rs' = B. \end{gathered} \tag{2}$$

The whole class of isospectral beams corresponding to the considered coefficients (1), can be obtained from the solutions of the system (2), as discussed in [2]. This is a system of nonlinear differential equations and it is not easy to find the explicit solution, but it is easy to see that a particular solution for this system is

$$\begin{gathered} r = -\beta - 2\gamma, \\ s = \gamma^2 + \beta\gamma - \beta' - \gamma'. \end{gathered} \tag{3}$$

Now it is natural to ask what are the other possible solutions?

*Mathematics Subject Classifications: 34L30, 53-XX.

†Department of Mathematics, Sahand University of Technology, Tabriz, Iran
‡Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran

Our purpose in this paper is to show that system (2) has no other solutions and so that the classification obtained in the earlier paper [2] is complete.

Jet space machinery is a well known tool in the geometric theory of the differential equations via prolongation method [4]. The application of this machinery to prove the uniqueness of a solution to the system of differential equations also seems to be interesting, although prolongation won't be used here. The crucial point in using this technique is related to the decision on jet coordinates.

In section 2 through an example equation we outline the method of this paper and express that the natural way of solving this type of equations requires jet space machinery. In section 3 we review needed facts from jet space theory. In section 4 it is the finite dimension of the applied jet space which helps us to prove the uniqueness.

2 Example Equation

Let $x = x(t)$ be a function of t, A is a known function of x and its derivatives, for example, $A(x) = 2x - x'$. To express the method of solution let us consider a first order differential equation

$$\Delta(r, r', A) = 2r - r' - A = 0 \tag{4}$$

where $r = r(x, x', x'', \cdots)$ is a function of x and its derivatives and where r' is stands for dr/dt. One may note that r as the dependent variable of (4) is a function of t via $x = x(t)$, and so dr/dt can be evaluated through the following series:

$$\frac{dr}{dt} = x' \frac{\partial r}{\partial x} + x'' \frac{\partial r}{\partial x'} + \cdots . \tag{5}$$

It is worth noting that (5) is not an infinite series, because r is a function of x and its time derivatives up to a finite order. To solve this differential equation (4) we use induction on n (= the highest derivative order of x inside the unknown function r). For $n = 0$, r is just a function of x, for $n = 1$ we have $r = r(x, x')$, for $n = 2$ then $r = r(x, x', x'')$ and so on.

As case 1, consider $n = 0$ and so $r = r(x)$ and $r' = x'r_x$. Then the above equation Δ in (4) reduces to $2r - x'r_x = 2x - x'$ or

$$(2r - 2x) - x'(r_x - 1) = 0 \tag{6}$$

where r_x stands for $\partial r/\partial x$.

We know r in this case is free of x' *i.e.* although r is a function of x but it is not any function of derivatives of x. So the coefficient of x' in (6) should put to zero. It is then deduced that $r_x = 1$ and finally $r = x$ is the solution to (4).

For case 2 we set $n = 1$ so $r = r(x, x')$ and $r' = x'r_x + x''r_{x'}$. Then the above equation Δ in (4) reduces to

$$(2r - 2x) - x'(r_x - 1) - x''r_{x'} = 0.$$

In this case we know that r is free of x'' so we deduce $r_{x'} = 0$, and this condition reduces this case 2 to be continued same as case 1.

As the final case of induction we assume that the procedure for $r = r(x, x'^{(n-1)})$ reduces to case 1, and try to extend the procedure for the case where we assume r is a function of x and its time derivatives up to order n, *i.e.* $r = r(x, x'^{(n)})$.

Computing $r' = x'r_x + x''r_{x'} + \cdots + x^{(n+1)}r_{x^{(n)}}$ by (5) and putting inside (4), one can see easily that assumed r is free of $x^{(n+1)}$ and so the coefficient of $x^{(n+1)}$ in (4), which is exactly $r_{x^{(n)}}$, should vanish. But then this case reduces to the previous case where r was not a function of $x^{(n)}$. By induction hypothesis it finally reduces to case $n = 0$ and we find that the only finite solution to (4) is $r(x) = x$. This completes our example and demonstrates the procedure of the paper.

The natural framework to compute total derivatives like (5) is Jet Space theory, which will be discussed shortly in the next section 3. Then in section 4 we apply the technique to the beam equation (2).

3 Jet Spaces

Jet spaces are the space of actions of the extended transformations (prolongations) of group of transformations admitted by a given system S of differential equations. Consider such a group of transformations as :

$$\begin{aligned} x^* &= X(x, u; \epsilon) \\ u^* &= U(x, u; \epsilon) \end{aligned} \tag{7}$$

acting on a space of $n + m$ variables

$$\begin{aligned} x &= (x_1, x_2, \ldots, x_n) \\ u &= (u^1, u^2, \ldots, u^m) \end{aligned}$$

where x corresponds to the n *independent variables* and u corresponds to the m *dependent variables* appearing in S. Assume that

$$u = F(x) = (f^1(x), f^2(x), ..., f^m(x))$$

denotes a solution of S.

Let $\underset{1}{u}$ denote the set of coordinates corresponding to all first order partial derivatives of u with respect to x:

$$\underset{1}{u} = \left(\frac{\partial u^1}{\partial x_1}, \frac{\partial u^1}{\partial x_2}, \ldots, \frac{\partial u^1}{\partial x_n}, \frac{\partial u^2}{\partial x_1}, \frac{\partial u^2}{\partial x_2}, \ldots, \frac{\partial u^2}{\partial x_n}, \ldots, \frac{\partial u^m}{\partial x_1}, \frac{\partial u^m}{\partial x_2}, \ldots, \frac{\partial u^m}{\partial x_n} \right);$$

$\underset{1}{u}$ has $n \times m$ coordinates. In general let $\underset{k}{u}$ denote the set of coordinates corresponding to all k-th order partial derivatives of u with respect to x, i.e. $u^\mu_{i_1 i_2 \cdots i_k} = \dfrac{\partial^k u^\mu}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_k}}$ where $i_j = 1, \ldots, n$, $j = 1, \ldots, k$, and $\mu = 1, \ldots, m$. Note that $\underset{k}{u}$ has $m \times \begin{pmatrix} n+k-1 \\ k \end{pmatrix}$ coordinates.

The Lie group of transformations (7) acting on (x, u)-space can now be “naturally" extended (prolongated) to act on $(x, u, \underset{1}{u}, \ldots, \underset{k}{u})$-space called k-jet space, [4, 1, 5, 3].

In a k-jet space, since we regard x, u, $\underset{1}{u}$, ..., and $\underset{k}{u}$ as coordinates of the space so, the partial derivatives of them with respect to each other are zero. Using this fact we define [4, 1] a very well known *total derivative operator* acting on jet spaces by:

$$\frac{D}{Dx_j} = \frac{\partial}{\partial x_j} + \sum_{\mu} u_j^{\mu} \frac{\partial}{\partial u^{\mu}} + \sum_{i,\mu} u_{ij}^{\mu} \frac{\partial}{\partial u_i^{\mu}} + \cdots \tag{8}$$

4 Modelling the Beam Equation in a Jet Space

We take a base space of $\{t, \beta, \gamma\}$, for this $\{t\}$ as independent variable and $\{\beta, \gamma\}$ as dependent variables. The jet coordinates will be $\{t, \beta, \gamma, \beta', \gamma', \beta'', \gamma'', ...\}$. To model the beam equation in this framework, we join the two equations of (2) by solving the first one to $s = \frac{1}{2}(A - r'^2)$ and insert it in the second one. Then we obtain a single third order differential equation on r:

$$2\,r''' - 6\,r\,r'' - 7(r'^2 + 4\,(A + 2\,r^2)r' + 2\,r\,A'^4 - 2Ar^2 - A^2 - 2A'' + 4B = 0. \tag{9}$$

Where r as stated earlier is a function of β and γ and their derivatives and A, B are given by (1). By the chosen base space, primes over r should be interpreted as the total derivatives on the jet space. So we compute total derivatives (8) of r as a function of jet variables with respect to t. Note that A, B, and r are function of t via $\beta(t)$ and $\gamma(t)$:

$$\begin{aligned} \tfrac{Dr}{Dt} &= \beta' r_\beta + \gamma' r_\gamma + \beta'' r_{\beta'} + \gamma'' r_{\gamma'} + \cdots \\ \tfrac{D^2 r}{Dt^2} &= \beta'' r_\beta + \gamma'' r_\gamma + \beta''' r_{\beta'} + \gamma''' r_{\gamma'} + \cdots \\ \tfrac{D^3 r}{Dt^3} &= \beta''' r_\beta + \gamma''' r_\gamma + \beta^{(4)} r_{\beta'} + \gamma^{(4)} r_{\gamma'} + \cdots \end{aligned} \tag{10}$$

Now we discuss some cases:

Case 1 We assume $r = r(\beta, \gamma)$. If we calculate (9) and rearrange, we find:

$$\left(-2 - 2\frac{\partial r}{\partial \beta}\right)\beta''' + \left(-4 - 2\frac{\partial r}{\partial \gamma}\right)\gamma''' + \cdots = 0. \tag{11}$$

By setting the coefficients of β''' and γ''' equal to zero, we find that $r_\beta = -1$ and $r_\gamma = -2$. Considering these conditions in (11), meaning that the other derivatives of r_β and r_γ are zero, we find the following equation:

$$\begin{aligned} 0 \;\; = \;\; & (-r - \beta - 2\gamma)\,\gamma'' + \left(2r^2 + r\beta - \beta^2 + 2r\gamma - 4\beta\gamma - 4\gamma^2\right)\gamma' \\ & + \left(\frac{1}{2}r^2 + r\beta + \frac{1}{2}\beta^2 + r\gamma + \beta\gamma\right)\beta' + \cdots . \end{aligned} \tag{12}$$

in this case the coefficient of γ'' should be zero, too. By setting $r = -\beta - 2\gamma$ all the other coefficients also must be zero and the equation fulfills. This proves that, as we knew previously,

$$r = -\beta - 2\gamma \tag{13}$$

is a solution to (9).

Case 2 We assume $r = r(\beta, \gamma, \beta', \gamma')$. If we calculate (9) and rearrange, we find:

$$\begin{aligned}
& \left(-2\frac{\partial r}{\partial \beta'}\right)\beta^{(4)} + \left(-2\,\frac{\partial r}{\partial \gamma'}\right)\gamma^{(4)} \\
& + \left(-2 - 2\frac{\partial r}{\partial \beta} + 6\,r\frac{\partial r}{\partial \beta'} - 6\beta'\frac{\partial^2 r}{\partial \gamma \partial \beta'} - 6\,\beta''\frac{\partial^2 r}{\partial \beta'^2} - 6\gamma''\frac{\partial^2 r}{\partial \beta' \partial \gamma'}\right)\beta''' \\
& + \left(-4 - 2\frac{\partial r}{\partial \gamma} + 6r\frac{\partial r}{\partial \gamma'} - 6\beta'\frac{\partial^2 r}{\partial \beta \partial \gamma'} - 6\gamma'\frac{\partial^2 r}{\partial \gamma \partial \gamma'} - 6\beta''\frac{\partial^2 r}{\partial \beta' \partial \gamma'} - 6\gamma''\frac{\partial^2 r}{\partial \gamma'^2}\right) \\
& + \cdots \\
= \;& 0
\end{aligned}$$

As we have assumed in this case that r is not a function of $\beta^{(4)}$ and $\gamma^{(4)}$, thus $\dfrac{\partial r}{\partial \beta'} = 0$ and $\dfrac{\partial r}{\partial \gamma'} = 0$. So r is just a function of β and γ and we should return to case 1 to continue.

Case 3 We assume $r = r(\beta, \gamma, \beta', \gamma', \cdots, \beta^{(n)}, \gamma^{(n)})$, where $n \geq 2$. Our goal in this case is to show that again it returns to the previous case 1 and so selecting greater n will reveal no other solution. To accomplish this it is sufficient to drop dependency of r on $\beta^{(n)}$ and $\gamma^{(n)}$. To proceed first we should find which term of (9) will produce the largest order of derivation:

$$\begin{aligned}
r'''\,\tilde{}\,\beta^{(n+3)}\frac{\partial r}{\partial \beta^{(n)}} + \gamma^{(n+3)}\frac{\partial r}{\partial \gamma^{(n)}} + \cdots \\
-6rr''\,\tilde{}\,r''\,\tilde{}\,\beta^{(n+2)}\frac{\partial r}{\partial \beta^{(n)}} + \gamma^{(n+2)}\frac{\partial r}{\partial \gamma^{(n)}} + \cdots \\
-7\,(r')^2\,\tilde{}\,\beta^{(n+1)}\beta^{(n+1)}\frac{\partial r}{\partial \beta^{(n)}}\frac{\partial r}{\partial \beta^{(n)}} + \beta^{(n+1)}\gamma^{(n+1)}\frac{\partial r}{\partial \beta^{(n)}}\frac{\partial r}{\partial \gamma^{(n)}} + \cdots
\end{aligned} \tag{14}$$

Calculations in (14) show that replacing r of this case in (???) will produce $\beta^{(n+3)}$ and $\gamma^{(n+3)}$ as the greatest derivative. Rearranging (14) will be similar to:

$$\begin{aligned}
0 \;=\; & \beta^{(n+3)}\frac{\partial r}{\partial \beta^{(n)}} + \gamma^{(n+3)}\frac{\partial r}{\partial \gamma^{(n)}} \\
& + \Delta(r, r_\beta, r_\gamma, r_{\beta'}, r_{\gamma'}, \cdots, r_{\beta^{(n)}}, r_{\beta^{(n)}}, \beta, \gamma, \beta', \gamma', ..., \beta^{(n+2)}, \gamma^{(n+2)})
\end{aligned} \tag{15}$$

In order to make (15) true, as r is a function of jet coordinates up to $\beta^{(n)}$ and $\gamma^{(n)}$, we see coefficients of $\beta^{(n+3)}$ and $\beta^{(n+3)}$ should vanish. This is possible only when r has no dependency on $\beta^{(n)}$ and $\gamma^{(n)}$.

Summarizing this section, indeed we have proved the following theorem.

THEOREM 1. If r satisfies in the equation (9) and r is a function of β and γ and their derivatives, then the equation (9) has the unique solution $r = -\beta - 2\gamma$.

References

[1] G. W. Bluman and S. Kumei, Symmetries and Differential Equations, Springer-Verlag, 1989.

[2] K. Ghanbari, On the isospectral beams, Electronic Journal of Differential Equations Conference, 12(2005) 57–64.

[3] S. Lie, Vorlesungen über Differentialgleichungen mit Bekannten Infinitesimalen Transformationen, Leipzig, 1891, (Reprinted by Chelsea Publishing, New York 1967).

[4] P. J. Olver, Applications of Lie groups to Differential Equations, Springer-Verlag, 1993.

[5] L. V. Ovsiannikov, Group analysis of Differential Equations, Nauka, Moscow, 1978, (English translation, W. F. Ames (Ed.), Academic Press, New York 1982).