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Polynomial Solutions Of A Generalization Of The First Painlevé Differential Equation*

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Abstract

In this paper we consider a generalization of the first Painlevé differential equation. We show that all its polynomial solutions can be computed in a systematic manner.

1 Introduction

Paul Painlevé in his lectures delivered in Stockholm [4] defined the first Painlevé differential equation as

$$y''(z) = 6y^2(z) + z,\ z \in \mathbf{C}$$

which is important in several domains of mathematics and physics.

In this paper, we are concerned with one type of generalization of the first Painlevé differential equation, namely, the following 'second order algebraic differential equation',

$$P_3(z)y''(z) = P_2(z)y^2(z) + P_1(z)y(z) + P_0(z),\ z \in \mathbf{C}, \tag{1}$$

where $\{P_0, P_1, P_2, P_3\}$ is a set of polynomials defined over the complex plane C such that P_3 and P_2 are nontrivial. We will set $p_i = \deg P_i$ for $i = 0, 1, 2, 3$. In case P_i is trivial, we define $\deg P_i = -\infty$. As usual, we adopt the convention that $\max\{-\infty, p\} = p$ for any real number p.

We will show that equation (1) has only a finite number of polynomial solutions and they can be computed in a systematic manner. We remark that such results are not true for every second order algebraic differential equation. For instance, for each nonnegative integer, the polynomial $y(z) = z^n$ satisfies the second order equation $zyy'' = z(y')^2 - yy'$.

There are now a lot of information on finding exact solutions of differential equations. However, the simplest exact solutions are naturally the polynomials. For general information, see e.g. [3], while for first order algebraic differential equations, one may consult [1, 2].

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2 Main Results

We first write $P_i = P_i(z)$ with degree p_i in the form

$$P_i(z) = P_{p_i}^{(i)} z^{p_i} + P_{p_i-1}^{(i)} z^{p_i-1} + \cdots + P_1^{(i)} z + P_0^{(i)}, \quad i = 0, 1, 2, 3, \tag{2}$$

where

$$P_{p_i}^{(i)} \neq 0, i = 2, 3.$$

It is easy to determine the set of all polynomials solutions of (1) with degree less than or equal to 1. Indeed, we simply substitute $y(z) = y_1 z + y_0$ into (1) and find

$$P_2(z)(y_1 z + y_0)^2 + P_1(z)(y_1 z + y_0) + P_0(z) = 0, \ z \in \mathbf{C}.$$

After expansion, we may find a polynomial in z with coefficients involving algebraic expressions of y_0 and y_1. Equating each of these expressions to 0 then yield a set of nonlinear equations in y_0 and y_1, which can in principle yields all possible solutions of y_0 and y_1. As an alternate approach, we may also put

$$\Phi(z) = P_2(z) y^2 + P_1(z) y + P_0(z), \ z \in \mathbf{C},$$

and

$$\Delta(z) = P_1^2(z) - 4P_2(z)P_0(z), \quad z \in \mathbf{C}.$$

Then we can write, successively,

$$\begin{aligned} \Phi(z) &= P_2(z)\left(y^2 + \frac{P_1(z)}{P_2(z)}y + \frac{P_0(z)}{P_2(z)}\right) \\ &= P_2(z)\left(\left(y + \frac{P_1(z)}{2P_2(z)}\right)^2 - \frac{\Delta(z)}{4P_2^2(z)}\right). \end{aligned}$$

Hence $\Phi(z) = 0$ if, and only if,

$$y = -\frac{P_1(z)}{2P_2(z)} + \frac{\pm\sqrt{\Delta(z)}}{2P_2(z)}.$$

If $\Delta(z) \neq P^2(z)$ for any polynomial $P(z)$, then there cannot be any polynomial solutions with degree ≤ 1. If $\Delta(z) = P^2(z)$ for some polynomial $P(z)$, then

$$y = \frac{-P_1(z) \pm P(z)}{2P_2(z)};$$

and if at least one of $\frac{-P_1(z) \pm P(z)}{2P_2(z)}$ is a polynomial of degree ≤ 1, then (1) admits at most two polynomial solutions of degree ≤ 1.

As an example, let us consider

$$(z^2 + 1)y''(z) = y^2 + (1 - z)y - 2z^2 + z, \ z \in \mathbf{C},$$

and let us try to find its polynomial solutions of degree ≤ 1. If $y = y_1 z + y_0$, we have

$$\Phi(z) = z^2 y_1^2 - z^2 y_1 - 2z^2 + 2z y_0 y_1 - z y_0 + z y_1 + z + y_0^2 + y_0,$$

so that $\Phi(z) = 0$ if, and only if,

$$\begin{cases} y_1^2 - y_1 - 2 = 0, \\ 2y_0 y_1 - y_0 + y_1 + 1 = 0, \\ y_0^2 + y_0 = 0. \end{cases}$$

From the third equation we see that $y_0 = 0$ or $y_0 = -1$. In case $y_0 = 0$, the second equation gives $y_1 = -1$, and equation one is also satisfied by such a y_1. Hence $y(z) = -z$. In the case where $y_0 = -1$, the second equation gives $y_1 = 2$, and equation one is also satisfied. Hence $y(z) = 2z - 1$.

If we use the 'discriminant' $\Delta(z) = (1-z)^2 - 4(-2z^2 + z) = (3z-1)^2$, then $P(z) = 3z - 1$, and hence

$$y = \frac{-P_1(z) \pm P(z)}{2P_2(z)} = \frac{-(1-z) \pm (3z-1)}{2},$$

which is just

$$y = -z \text{ or } 2z - 1$$

as before.

Next, we seek polynomial solutions with degree ≥ 2. First, note that if $y = y(z)$ is a polynomial solution of (1) with degree $n \geq 2$, then $\deg\left(P_i y^i\right) = p_i + in$ for $i = 0, 1, 2$ and $\deg\left(P_3 y''\right) = p_3 + n - 2$. This motivates us to define 4 indices $\kappa_0, \kappa_1, \kappa_2, \kappa_3$ associated with $\{P_0, P_1, P_2, P_3\}$: for each $i \in \{0, 1, 2, 3\}$, if $P_i \neq 0$, let $\kappa_i = \kappa_i(n)$, be defined for each $n \in \{2, 3, ...\}$ by

$$\kappa_i(n) = \begin{cases} p_i + in & i = 0, 1, 2, \\ p_3 + n - 2 & i = 3. \end{cases}$$

and we take $\kappa_i(n) = -\infty$ if $P_i(z) \equiv 0$.

We will also set

$$\kappa(n) = \max\left\{\kappa_0(n), ..., \kappa_3(n)\right\}, \ n = 2, 3, 4, ... \ .$$

A necessary condition for the existence polynomial solution of degree greater then or equal to 2 is as follows.

LEMMA 1. If $y = y(z)$ is a polynomial solution of (1) with degree $n \geq 2$, then there exist $t, j \in \{0, 1, 2, 3\}$ such that $t < j$ and

$$\kappa_t(n) = \kappa_j(n) \geq \kappa_s(n), \ \forall s \in \{0, 1, 2, 3\}. \tag{3}$$

PROOF. Let

$$y(z) = y_n z^n + y_{n-1} z^{n-1} + \cdots + y_1 z + y_0, \ y_n \neq 0, \tag{4}$$

be a polynomial solution of (1) with degree $n \geq 2$. Then $\deg\left(P_i y^i\right) = \kappa_i(n)$ for $i = 0, 1, 2$ and $\deg\left(P_3 y''\right) = \kappa_3(n)$. Let t be the least positive integer such that $\kappa_t(n) = \kappa(n)$. By substituting $y = y(z)$ into (1), we see that

$$\begin{aligned} n(n-1)y_n P^{(3)}_{p_3} z^{\kappa_3(n)} + \cdots &= \left\{ y_n^2 P^{(2)}_{p_2} z^{\kappa_2(n)} + \cdots \right\} \\ &\quad + \left\{ y_n P^{(1)}_{p_1} z^{\kappa_1(n)} + \cdots \right\} + \left\{ P^{(0)}_{p_0} z^{\kappa_0(n)} + \cdots \right\} \end{aligned}$$

for $z \in \mathbf{C}$. Hence, if $\kappa_t(n) > \kappa_j(n)$ for $j \neq t$, then $P^{(t)}_{p_t} y_n^t = 0$, which is contrary to our assumption. Thus there is some $j > t$ such that $\kappa_j(n) = \kappa_t(n) \geq \kappa_s(n)$. The proof is complete.

We say that a positive integer n is $\{P_0, P_1, P_2, P_3\}$-feasible (or feasible if no confusion is caused) if the indices $\kappa_0, ..., \kappa_3$ associated with $\{P_0, P_1, P_2, P_3\}$ satisfy (3) for some $t, j \in \{0, 1, 2, 3\}$ with $t < j$.

LEMMA 2. The set of feasible integers are bounded from above.

PROOF. Since

$$\begin{aligned} \kappa(j) &= \max\{p_0, p_1 + j, p_2 + 2j, p_3 + j - 2\} \\ &= p_2 + 2j = \kappa_2(j) > \max\{\kappa_0(j), \kappa_1(j), \kappa_3(j)\} \end{aligned}$$

for all sufficiently large j, we may let J be the first positive integer such that the above chain of (equalities and) inequalities hold for all $j \geq J$. In view of Lemma 1, a feasible integer n must be less than J so that $n \leq J - 1$. The proof is complete.

Once we have determined an upper bound for n, we may determine the set of feasible integers by checking whether $\max\{\kappa_0(n), ..., \kappa_3(n)\}$ is attained by at least two members. Next, let n be such a feasible integer. We will try to look for polynomial solutions of the form

$$y(z) = y_n z^n + W(z), \ y_n \neq 0, \tag{5}$$

where $W(z) = y_{n-1} z^{n-1} + \cdots + y_1 z + y_0$. By substituting $y(z)$ into (1) and then rearranging the resulting equation, we obtain a polynomial equation

$$H_{\kappa(n)}(y_n) z^{\kappa(n)} + \cdots = 0, \ z \in \mathbf{C},$$

where $H_{\kappa(n)}$ is a polynomial in y_n with degree ≤ 2. By comparing coefficients, we see that

$$H_{\kappa(n)}(y_n) = 0. \tag{6}$$

Three cases can then occur: (i) $H_{\kappa(n)}$ is trivial, (ii) $\deg H_{\kappa(n)} = 0$ but $H_{\kappa(n)}$ is nontrivial, and (iii) $\deg H_{\kappa(n)} \geq 1$.

The case (ii) is easy to deal with. Indeed, this case leads to a nonzero constant equals zero. In other words, there is no solution for (6) and hence no polynomial solution (with degree n) for (1).

If case (iii) holds, we may then find at least one and at most 2 solutions of (6). Let y_n be such a solution, then in view of (6) and

$$P_3(z)\left(y_n z^n + W(z)\right)'' = P_2(z)\left(y_n z^n + W(z)\right)^2 + P_1(z)\left(y_n z^n + W(z)\right) + P_0(z),$$

we see that W is a polynomial solution of

$$P_3(z)W'' = P_2(z)W^2 + G_1(z)W + G_0(z),\ z \in \mathbf{C}, \tag{7}$$

for some polynomials G_0, G_1, and the degree of W is $\leq n-1$. Since (7) is of the form (1), we may start a new recursion process by replacing $\{P_0, P_1, P_2, P_3\}$ in (1) with $\{G_0, G_1, P_2, P_3\}$ and looking for polynomial solutions of the form $W(z)$ (with degree $n-1$).

The case (i) is more difficult. Let n be a feasible integer. Assume that

$$y(z) = y_n z^n + y_{n-1}z^{n-1} + \cdots + y_1 z + y_0,\ z \in \mathbf{C},\ n \geq 2,\ y_n \neq 0, \tag{8}$$

is a polynomial solution of (1). If $H_{\kappa(n)}$ in (6) is trivial, we assert that

$$\kappa(n) = \kappa_3(n) = \kappa_1(n) > \kappa_t(n),\ t = 0, 2 \tag{9}$$

and

$$n(n-1)P_{p_3}^{(3)} = P_{p_1}^{(1)}. \tag{10}$$

Indeed, if $\kappa(n) = \kappa_0(n) > \kappa_t(n)$ for $t \neq 0$, then $H_{\kappa(n)}(y_n)$ is equal to $P_{p_0}^{(0)}$ plus terms with higher powers of y_n; if $\kappa(n) = \kappa_1(n) > \kappa_t(n)$ for $t \neq 1$, then $H_{\kappa(n)}(y_n) = P_{p_1}^{(1)} y_n$; and if $\kappa(n) = \kappa_2(n) > \kappa_t(n)$ for $t \neq 2$, then $H_{\kappa(n)}(y_n) = P_{p_2}^{(2)} y_n^2$. In these cases, $H_{\kappa(n)}$ is not trivial. There remains the only case where $\kappa(n) = \kappa_1(n) = \kappa_3(n) > \kappa_t(n)$ for $t \neq 0, 2$. Then $H_{\kappa(n)}(y_n) = n(n-1)P_{p_3}^{(3)} y_n - P_{p_1}^{(1)} y_n$, which shows that $H_{\kappa(n)}$ is trivial if and only if $n(n-1)P_{p_3}^{(3)} = P_{p_1}^{(1)}$.

Note that a direct consequence of (9) is that

$$p_3 - 2 = p_1,\ n < p_3 - p_2 - 2,\ p_0 < p_3 + n - 2. \tag{11}$$

Substituting (8), (2) and

$$P^{(1)}(z) = n(n-1)P_{p_3}^{(3)} z^{p_3 - 2} + \sum_{i=0}^{p_3 - 3} P_i^{(1)} z^i$$

into (1) and then rearranging the resulting equation, we obtain

$$\begin{aligned} & \left(\sum_{i=0}^{n-1} (i(i-1) - n(n-1)) y_i P_{p_3}^{(3)} z^{p_3 + i - 2}\right) \\ = \quad & \left(\sum_{i=0}^{p_2} P_i^{(2)} z^i\right)\left(\sum_{i=0}^{n} y_i z^i\right)^2 + \left(\sum_{i=0}^{p_1 - 1} P_i^{(1)} z^i\right)\left(\sum_{i=0}^{n} y_i z^i\right) \\ & - \left(\sum_{i=0}^{p_3 - 1} P_i^{(3)} z^i\right)\left(\sum_{i=2}^{n} i(i-1) y_i z^{i-2}\right) + \sum_{i=0}^{p_0} P_i^{(0)} z^i \end{aligned} \tag{12}$$

for all $z \in \mathbf{C}$. By comparing coefficients, we obtain the following system of $p_3 + n - 2$ equations:

$$[(n-1)(n-2) - n(n-1)]P_{p_3}^{(3)} y_{n-1} \quad = \quad R_1(y_n, y_{n-1}, ..., y_1, y_0),$$

$$\begin{aligned}
[(n-2)(n-3)-n(n-1)]P_{p_3}^{(3)}y_{n-2} &= R_2(y_n,y_{n-1},...,y_1,y_0),\\
\cdots &= \cdots\\
[(n-i)(n-i-1)-n(n-1)]P_{p_3}^{(3)}y_{n-i} &= R_i(y_n,y_{n-1},...,y_1,y_0),\\
\cdots &= \cdots\\
-n(n-1)P_{p_3}^{(3)}y_1 &= R_{n-1}(y_n,y_{n-1},...,y_1,y_0),\\
-n(n-1)P_{p_3}^{(3)}y_0 &= R_n(y_n,y_{n-1},...,y_1,y_0),\\
V_1(y_n,y_{n-1},...,y_1,y_0) &= 0,\\
\cdots &= \cdots,\\
V_{p_3-2}(y_n,y_{n-1},...,y_1,y_0) &= 0,
\end{aligned}$$

where $R_1,...,R_n,V_1,...,V_{p_3-2}$ are polynomials.

We first show that for each $i\in\{1,2,...,n\}$, R_i is independent of $y_{n-i},y_{n-i-1},...,y_0$, that is, $R_i=R_i(y_n,...,y_{n-i+1})$. To see this, we need the elementary fact that if we expand the polynomial $\left(\sum_{i=0}^n y_iz^i\right)^2$ into a sum of separate terms, then the term that contains y_t, where $t\in\{0,1,...,n-1\}$, and the highest power of z is $2y_ty_nz^{n+t}$. Now suppose to the contrary that there exists an integer $t\in\{0,1,...,n-i\}$ such that R_i depends on y_t. Then there are three cases. First, if y_t arises from expanding $P_2(z)\left(\sum_{i=0}^n y_iz^i\right)^2$, then $2y_ty_nP_{p_2}^{(2)}z^{n+t+p_2}$ is the term with the highest power of z. The i-th equation of the above system arises from the coefficients of the term $z^{p_3+n-i-2}$ in the equation (12). Since $n+t+p_2\geq p_3+n-i-2$ and $t\leq n-i$, we must have $n\geq p_3-p_2-2$, which is contrary to (11). Second, if y_t arises from expanding $\left(\sum_{i=0}^{p_1-1}P_i^{(1)}z^i\right)\left(\sum_{i=0}^n y_iz^i\right)$, then $P_{p_1-1}^{(1)}y_tz^{p_1-1+t}$ is the term with the highest power of z. Again, since $p_1-1+t\geq p_3+n-i-2$ and $t\leq n-i$, we must have $n-i\geq n-i+1$, which is impossible. Finally, if y_t arises from expanding $\left(\sum\limits_{i=0}^{p_3-1}P_i^{(3)}z^i\right)\left(\sum\limits_{i=2}^{n}i(i-1)y_iz^{i-2}\right)$, then $P_{p_3-1}^{(3)}t(t-1)y_tz^{p_3+t-3}$ is the term with the highest power of z. Since $p_3+t-3\geq p_3+n-i-2$ and $t\leq n-i$, we must have $n-i\geq n-i+1$, which is impossible. The proof of our assertion is complete.

We may now rewrite the above system in the form

$$\begin{aligned}
[(n-1)(n-2)-n(n-1)]P_{p_3}^{(3)}y_{n-1} &= R_1(y_n),\\
[(n-2)(n-3)-n(n-1)]P_{p_3}^{(3)}y_{n-2} &= R_2(y_n,y_{n-1}),\\
\cdots &= \cdots\\
[(n-i)(n-i-1)-n(n-1)]P_{p_3}^{(3)}y_{n-i} &= R_i(y_n,y_{n-1},...,y_{n-i+1}),\\
\cdots &= \cdots\\
-n(n-1)P_{p_3}^{(3)}y_1 &= R_{n-1}(y_n,y_{n-1},...,y_2),\\
-n(n-1)P_{p_3}^{(3)}y_0 &= R_n(y_n,y_{n-1},...,y_1),\\
V_1(y_n,y_{n-1},...,y_1,y_0) &= 0,\\
\cdots &= \cdots,\\
V_{p_3-2}(y_n,y_{n-1},...,y_1,y_0) &= 0,
\end{aligned}$$

Clearly, we may then express $y_{n-1}, y_{n-2}, ..., y_0$ recursively in terms of y_n, say,

$$y_i = F_{n-i}(y_n),\ i = 0, 1, ..., n-1,$$

and then substitute them into $V_1, ..., V_{p_3-2}$ to obtain

$$G_i(y_n) = V_i\left(y_n, F_1(y_n), ..., F_n(y_n)\right) = 0,\ i = 1, 2, ..., p_3 - 2. \tag{13}$$

We assert that the polynomials $G_1, G_2, ..., G_{p_3-2}$ cannot be trivial simultaneously. Suppose to the contrary that $G_1, G_2, ..., G_{p_3-2} \equiv 0$. Then

$$y(z) = y_n z^n + F_1(y_n) z^{n-1} + \cdots + F_n(y_n),\ z \in \mathbf{C},$$

is a solution for any $y_n \in \mathbf{C}$. Let us write

$$y(z) = F_0(y_n) z^n + F_1(y_n) z^{n-1} + \cdots + F_n(y_n),\ z \in \mathbf{C},$$

where F_0 is the identity polynomial. Let $h_i = \deg F_i$ for $i \in \{0, 1, ..., n\}$ and $h = \max\{h_0, h_1, ..., h_n\}$ (which is greater than or equal to 1 because $\deg F_0 = 1$). Let $z_0 \in \mathbf{C}$ such that $\deg y(z_0) = h$ and $P_2(z) \neq 0$ ($y(z_0)$ is considered as a polynomial in y_n). In view of (12),

$$\begin{aligned} &\left(\sum_{i=0}^{n-1} (i(i-1) - n(n-1)) y_i P_{p_3}^{(3)} z_0^{p_3+i-2}\right) \\ = \quad &\left(\sum_{i=0}^{p_2} P_i^{(2)} z_0^i\right)\left(\sum_{i=0}^{n} y_i z_0^i\right)^2 + \left(\sum_{i=0}^{p_1-1} P_i^{(1)} z_0^i\right)\left(\sum_{i=0}^{n} y_i z_0^i\right) \\ &- \left(\sum_{i=0}^{p_3-1} P_i^{(3)} z_0^i\right)\left(\sum_{i=2}^{n} i(i-1) y_i z_0^{i-2}\right) + \sum_{i=0}^{p_0} P_i^{(0)} z_0^i \end{aligned} \tag{14}$$

for $z \in \mathbf{C}$. However, this is impossible since

$$\deg\left(\sum_{i=0}^{n-1} (i(i-1) - n(n-1)) y_i P_{p_3}^{(3)} z_0^{p_3+i-2}\right) \le h < 2h,$$

$$\deg\left(\sum_{i=0}^{p_2} P_i^{(2)} z_0^i\right)\left(\sum_{i=0}^{n} y_i z_0^i\right)^2 = 2h,$$

$$\deg\left(\sum_{i=0}^{p_1-1} P_i^{(1)} z_0^i\right)\left(\sum_{i=0}^{n} y_i z_0^i\right) \le h < 2h$$

and

$$\deg\left(\sum_{i=0}^{p_3-1} P_i^{(3)} z_0^i\right)\left(\sum_{i=2}^{n} i(i-1) y_i z_0^{i-2}\right) \le h < 2h.$$

The proof of our assertion is complete.

We may now summarize the above as follows. If n is feasible, a polynomial solution of the form (5) is said to be $\{P_0, P_1, P_2, P_3\}$ degenerate if (6) holds.

LEMMA 3. If a solution y of the form (8) is a $\{P_0, P_1, P_2, P_3\}$ degenerate polynomial, then there exist polynomials $F_1, ..., F_n$ such that $y_i = F_{n-i}(y_n)$ for $i = 0, 1, ..., n-1$, and polynomials $V_1, ..., V_{p_3-2}$ such that $V_i(y_n, F_1(y_n), ..., F_n(y_n)) = 0$ for $i = 0, 1, ..., n-1$. Furthermore, the polynomials $G_1, ..., G_{p_m-1}$ defined by $G_i(z) = V_i(z, F_1(z), ..., F_n(z))$ for $i = 1, ..., p_3 - 2$ cannot be simultaneously trivial.

Once we have determined that y of the form (8) is $\{P_0, P_1, P_2, P_3\}$ degenerate, then as before, we may check if some G_i is a trivial constant polynomial. In such as case, y cannot be a solution of (1). Else, we may let G be the greatest common divisor of $G_1, ..., G_{p_3-2}$. Then y_n equals to one of the roots (if they exist) of G.

We may now summarize our previous discussions as follows.

THEOREM 1. Given polynomials P_0, P_1, P_2, P_3 where P_3 and P_2 are not trivial, the equation (1) has only a finite number of polynomial solutions, and they can be computed by the method of undetermined coefficients in a systematic manner.

3 Examples

We illustrate our previous results by means of several examples.

EXAMPLE 1. Consider

$$(z^2-1)y''(z) = z^2y^2 + (1+2z^2)y + z^2 - 1, z \in \mathbf{C}. \tag{15}$$

If $y(z) = y_1 z + y_0$ is a solution of (15), then

$$\Phi(z) = z^4y_1^2 + 2z^3y_0y_1 + 2z^3y_1 + z^2y_0^2 + 2z^2y_0 + z^2 + zy_1 + y_0 - 1,$$

so that $\Phi(z) = 0$ if and only if

$$\begin{cases} y_1^2 = 0, \\ 2y_0y_1 + 2y_1 = 0, \\ y_0^2 + 2y_0 + 1 = 0, \\ y_1 = 0, \\ y_0 - 1 = 0. \end{cases}$$

Since the third and the fifth equations are incompatible, there is no polynomial solution of degree ≤ 1.

The same conclusion can be seen by considering the 'discriminant'

$$\Delta(z) = P_1^2(z) - 4P_2(z)P_0(z) = (1+2z^2)^2 - 4z^2(z^2-1) = 8z^2 + 1$$

which cannot be expressed as $P^2(z)$.

EXAMPLE 2. Consider the equation

$$\left(z^5 - z^3\right) y''(x) = 2zy^2 - (2z+1)y + z^2, z \in \mathbf{C}. \tag{16}$$

Since

$$\Delta(z) = (2z+1)^2 - 8zz^2 = -8z^3 + 4z^2 + 4z + 1$$

is not equal to any polynomial $P^2(z)$, we see that (16) has no polynomial solutions of degree ≤ 1.

The set of feasible integers associated to (16) is $\{2\}$ and $\kappa(2) = 5$. Let $y(z) = y_2 z^2 + y_1 z + y_0$, where $y_0, y_1, y_2 \in \mathbf{C}$, be a candidate solution of (16). Since

$$H_5(y_2) = 2y_2 - 2y_2^2 = 2y_2(1 - y_2)$$

implies that $y_2 = 0$ or $y_2 = 1$ and since (16) has no polynomial solutions of degree ≤ 1, we see that $y_2 = 1$. We put $W(z) = y(z) - z^2$. Then $\deg W \leq 1$. Furthermore,

$$\left(z^5 - z^3\right)(W''(z) + 2) = 2z(W + z^2)^2 - (2z+1)(W + z^2) + z^2,$$

$$2z(W + z^2)^2 - (2z+1)(W + z^2) + z^2 - \left(z^5 - z^3\right)(2) = W\left(2zW - 2z + 4z^3 - 1\right),$$

$$\left(z^5 - z^3\right)W''(z) = 2zW^2(z) + (4z^3 - 2z - 1)W(z).$$

But $\deg W \leq 1$. Thus the last equation is equivalent to

$$2zW^2(z) + (4z^3 - 2z - 1)W(z) = 0.$$

Hence $W(z) = 0$ or $W(z) = -\dfrac{4z^3 - 2z - 1}{2z}$. The latter function is not a polynomial, and hence $y(z) = z^2$ is the unique polynomial solution of (16).

EXAMPLE 3. Consider the equation

$$z^6 y''(x) = y^2 + 6z^4 y - z^6, z \in \mathbf{C}. \tag{17}$$

Here $\Delta(z) = (6z^4)^2 + 4z^6 = 4z^6\left(9z^2 + 1\right)$ which is not equal to any square of a polynomial $P(z)$. Then (17) has no polynomial solutions of degree ≤ 1.

The set of feasible integers associated to (17) is $\{4, 3, 2\}$ and $\kappa(4) = 8$. Let $y(z) = y_4 z^4 + y_3 z^3 + y_2 z^2 + y_1 z + y_0$, where $y_0, y_1, y_2, y_3, y_4 \in \mathbf{C}$, be a candidate solution of (17). Since

$$H_8(y_4) = 6y_4 - y_4^2 = 0,$$

we see that $y_4 = 6$ or $y_4 = 0$.

Suppose first that $y_4 = 6$. We put $W(z) = y(z) - 6z^4$. Then $\deg W \leq 3$, so that

$$z^6 W'' = W^2 + 18z^4 W - z^6, z \in \mathbf{C}. \tag{18}$$

The set of feasible integers less than or equal to 3 and associated to (18) is $\{3, 2\}$ and $\kappa(3) = 7$. Since

$$H_7(y_3) = -12y_3 = 0,$$

we see that $y_3 = 0$.

The set of feasible integers less than or equal to 2 and associated to (18) is $\{2\}$ and $\kappa(2) = 6$. Since

$$H_6(y_2) = -16y_2 + 1 = 0,$$

we see that $y_2 = \dfrac{1}{16}$. We put $W_2(z) = W(z) - \dfrac{1}{16}z^2$. Then $\deg W_2 \leq 1$, so that

$$z^6 W_2'' = W_2^2 + (18z^4 + \frac{1}{8}z^2)W_2 + \frac{1}{256}z^4, z \in \mathbf{C}. \tag{19}$$

We look for polynomial solutions of (19) of degree ≤ 1. By substituting $W_2(z) = y_1 z + y_0$ in (19), we obtain the system

$$\begin{cases} 18y_1 = 0, \\ 18y_0 + \frac{1}{256} = 0, \\ \frac{1}{8}y_1 = 0, \\ \frac{1}{8}y_0 + y_1^2 = 0, \\ 2y_0 y_1 = 0, \\ y_0^2 = 0. \end{cases}$$

The second and the sixth equations are incompatible, thus there is no polynomial solution of degree ≤ 1.

Next we consider the case where $y_4 = 0$. The set of feasible integers ≤ 3 and associated to (17) is $\{3, 2\}$ and $\kappa(3) = 7$. Since

$$H_7(y_3) = 6y_3 - 6y_3 \equiv 0,$$

we are in the degenerate case. We substitute $y(z) = y_3 z^3 + y_2 z^2 + y_1 z + y_0$ into (17) and after expansion, we find

$$\begin{cases} -4y_2 - y_3^2 + 1 = 0, \\ -6y_1 - 2y_2 y_3 = 0, \\ -6y_0 - 2y_1 y_3 - y_2^2 = 0, \\ -2y_0 y_3 - 2y_1 y_2 = 0, \\ -2y_0 y_2 - y_1^2 = 0, \\ 2y_0 y_1 = 0, \\ -y_0^2 = 0. \end{cases}$$

From the first equation of the above system, we may express y_2 in terms of y_3, then substituting y_2 into the second equation, we may express y_1 in terms of y_3, and then y_0 in terms of y_3. Substituting y_0 and y_1 into the other equations, we may then obtain

$$\begin{cases} y_2 = \dfrac{1}{4}(-y_3^2 + 1), \\ y_1 = -\dfrac{1}{12}(-y_3^2 + 1)y_3, \\ y_0 = \frac{11}{288}y_3^4 - \frac{7}{144}y_3^2 + \frac{1}{96}, \\ q_1(y_3) = -\frac{5}{144}y_3^5 + \frac{1}{72}y_3^3 + \frac{1}{48}y_3 = 0, \\ q_2(y_{3)=} - \frac{7}{576}y_3^6 + \frac{17}{576}y_3^4 - \frac{13}{576}y_3^2 + \frac{1}{192} = 0, \\ q_3(y_3) = (-y_3^2 + 1)y_3(\frac{11}{288}y_3^4 - \frac{7}{144}y_3^2 + \frac{1}{96}) = 0, \\ q_4(y_3) = (\frac{11}{288}y_3^4 - \frac{7}{144}y_3^2 + \frac{1}{96})^2 = 0. \end{cases}$$

Note that the greatest common divisor q of the polynomials q_1, q_2, q_3 and q_4 is $q(z) = z^2 - 1$. Thus $(y_3, y_2, y_1, y_0) = (1, 0, 0, 0)$ and $(y_3, y_2, y_1, y_0) = (-1, 0, 0, 0)$ are the solutions of the above system. Thus $y(z) = z^3$ and $y(z) = -z^3$ are the only polynomial solutions of (17).

As our final remark, the condition that $P_2(z)$ is nontrivial cannot be removed in Theorem 1. Indeed, there are infinitely many polynomials of the form $y(z) = \lambda z^3$ that satisfy $z^2 y''(z) = 6y(z)$.

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References

[1] D. Behloul and S. S. Cheng, Computation of all polynomial solutions of a class of nonlinear differential equations, Computing, 77(2006), 163–177.

[2] D. Behloul and S. S. Cheng, Polynomial solutions of a class of algebraic differential equations with quadratic nonlinearities, Southeast Asian Bulletin of Mathematics, 33(2009), 1029–1040.

[3] Y. J. Ma and X. S. Gao, Polynomial solutions of algebraic differential equations. (English summary) Computer mathematics (Matsuyama, 2001), 92–101, Lecture Notes Ser. Comput., 9, World Sci. Publ., River Edge, NJ, 2001.

[4] P. Painlevé, Leçons sur la théorie analytique des équations différentielles, professées à Stockholm, Reprinted: Oeuvre de Paul Painlevé Vol.1.2.3 (1895) Hermann, Paris.