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Equivalence Between An Approximate Version Of Brouwer's Fixed Point Theorem And Sperner's Lemma: A Constructive Analysis*

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Abstract

It is widely believed that Sperner's lemma and Brouwer's fixed point theorem are equivalent. But in second order arithmetic ([5]), although Sperner's lemma is proved in RCA_0, Brouwer's fixed point theorem is not. Also in Bishop style constructive mathematics, although Sperner's lemma can be constructively proved, Brouwer's fixed point theorem can not be constructively proved. We consider an approximate (or a constructive) version of Brouwer's fixed point theorem, and show the equivalence between Sperner's lemma and an approximate version of Brouwer's fixed point theorem. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

1 Introduction

Brouwer's fixed point theorem is extensively applied in economic theory and game theory. It is widely believed that Sperner's lemma and Brouwer's fixed point theorem are equivalent. But in second order arithmetic ([5]), although Sperner's lemma is proved in RCA_0, Brouwer's fixed point theorem is not. Also in Bishop style constructive mathematics, although Sperner's lemma can be constructively proved, Brouwer's fixed point theorem can not be constructively proved. Recently some authors have presented an approximate (or a constructive) version of Brouwer's fixed point theorem using Sperner's lemma. See [4] and [8]. We will show that Sperner's lemma and an approximate version of Brouwer's fixed point theorem are equivalent. We follow the Bishop style constructive mathematics according to [1], [2] and [3].

2 Approximate Version of Brouwer's Theorem

Let $\mathbf{p} = (p_0, p_1, \ldots, p_n)$ be a point in an n-dimensional simplex Δ, and consider a function φ from Δ to itself. n is a finite natural number. We define uniform continuity and approximate fixed point as follows.

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Uniform continuity of functions A function φ is uniformly continuous if for any $\mathbf{p}$, $\mathbf{p}'$ and $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\text{If } |\mathbf{p}' - \mathbf{p}| < \delta, \text{ then } |\varphi(\mathbf{p}') - \varphi(\mathbf{p})| < \varepsilon.$$

Approximate fixed point For each ε $\mathbf{p}$ is an approximate (or an ε-approximate) fixed point of φ if

$$|\mathbf{p} - \varphi(\mathbf{p})| < \varepsilon.$$

An approximate version of Brouwer's fixed point theorem is as follows.

THEOREM 1. (Approximate version of Brouwer's fixed point theorem) For each $\varepsilon > 0$ any uniformly continuous function from an n-dimensional simplex Δ to itself has an approximate fixed point.

PROOF. See [4] or Theorem 6 in [8][1].

3 From Approximate Version of Brouwer's Theorem to Sperner's Lemma

Let us partition the simplex. Figure 1 is an example of partition (triangulation) of a 2-dimensional simplex. In a 2-dimensional case we divide each side of Δ in m equal segments, and draw the lines parallel to the sides of Δ. Then, the 2-dimensional simplex is partitioned into m^2 triangles. We consider partition of Δ inductively for cases of higher dimension. In a 3 dimensional case each face of Δ is a 2-dimensional simplex, and so it is partitioned into m^2 triangles in the above mentioned way, and draw the planes parallel to the faces of Δ. Then, the 3-dimensional simplex is partitioned into m^3 trigonal pyramids. And similarly for cases of higher dimension.

Let K denote the set of small n-dimensional simplices of Δ constructed by partition. Vertices of these small simplices of K are labeled with the numbers $0, 1, 2, \dots, n$ subject to the following rules.

1. The vertices of Δ are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with 2, $\dots$, a point $(0, \dots, 0, 1)$ with n. That is, a vertex whose k-th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k.

2. If a vertex of K is contained in an $n-1$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of a vertex of that face.

3. If a vertex of K is contained in an $n-2$-dimensional face of Δ, then this vertex is labeled with some number which is the same as the number of a vertex of that face. And similarly for cases of lower dimension.

[1] [4] and [8] have shown the theorem for only a 2-dimensional case. But it can be extended to a general n-dimensional case. In [6] and [7] we have constructively shown an approximate version of Brouwer's fixed point theorem for an n-dimensional simplex and other fixed point theorems.

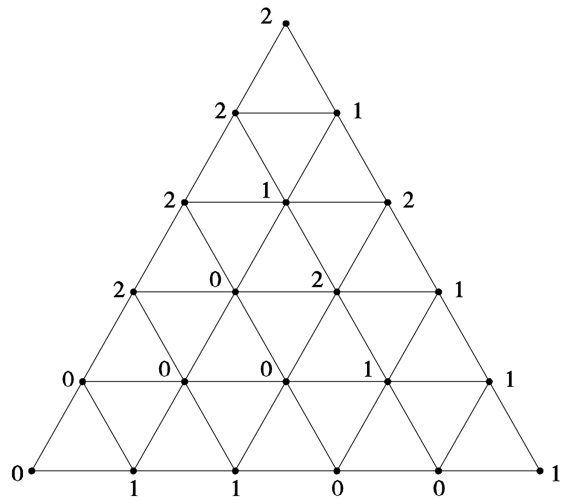

Figure 1: Partition and labeling of 2-dimensioanl simplex

4. A vertex contained in inside of Δ is labeled with an arbitrary number among 0, 1, $\ldots$, n.

A small simplex of K which is labeled with the numbers 0, 1, $\ldots$, n is called a *fully labeled simplex*. Sperner's lemma is as follows;

LEMMA 1. (Sperner's lemma) If we label the vertices of K following above rules 1 $\sim$ 4, then there exists at least one fully labeled simplex.

Now we show the following result.

THEOREM 2. Sperner's lemma is derived from the approximate version of Brouwer's fixed point theorem.

PROOF. Denote vertices of an n-dimensional simplex of K by $x^0, x^1, \ldots, x^n$, and denote the j-th component of x^i by x^i_j. These vertices are labeled according to the above rules 1 $\sim$ 4. Denote the label of x^i by $l(x^i)$. Let τ be a positive number which is smaller than $x^i_{l(x^i)}$ for all x^i, and define a function $f(x^i)$ as follows[2];

$$f(x^i) = (f_0(x^i), f_1(x^i), \ldots, f_n(x^i)),$$

and

$$f_j(x^i) = \begin{cases} x^i_j - \tau & \text{for } j = l(x^i), \\ x^i_j + \frac{\tau}{n} & \text{for } j \neq l(x^i). \end{cases} \tag{1}$$

f_j denotes the j-th component of f. From the labeling rules $x^i_{l(x^i)} > 0$ for all x^i, and so $\tau > 0$ is well defined. Since $\sum_{j=0}^n f_j(x^i) = \sum_{j=0}^n x^i_j = 1$, we have

$$f(x^i) \in \Delta.$$

[2] We refer to [9] about the definition of this function.

We extend f to all points in the simplex by convex combinations of its values on the vertices of the simplex. Let z be a point in the n-dimensional simplex of K whose vertices are $x^0, x^1, \dots, x^n$. Then, z and $f(z)$ are represented as follows;

$$z = \sum_{i=0}^{n} \lambda_i x^i, \text{ and } f(z) = \sum_{i=0}^{n} \lambda_i f(x^i),\ \lambda_i \geq 0,\ \sum_{i=0}^{n} \lambda_i = 1.$$

Let us show that f is uniformly continuous. Let z and z' be distinct points in the same small n-dimensional simplex of K. They are represented as

$$z = \sum_{i=0}^{n} \lambda_i x^i,\ z' = \sum_{i=0}^{n} \lambda'_i x^i,$$

and so

$$z - z' = \sum_{i=0}^{n} (\lambda_i - \lambda'_i) x^i \text{ and } z_j - z'_j = \sum_{i=0}^{n} (\lambda_i - \lambda'_i) x^i_j \text{ for each } j.$$

Then, we have

$$f(z) - f(z') = \sum_{i=0}^{n} (\lambda_i - \lambda'_i) f(x^i)$$

and for each j

$$\begin{aligned} f_j(z) - f_j(z') &= \sum_{i=0}^{n} (\lambda_i - \lambda'_i) x^i_j + \sum_{i:j\neq l(i)} (\lambda_i - \lambda'_i) \frac{\tau}{n} - \sum_{i:j=l(i)} (\lambda_i - \lambda'_i)\tau \\ &= z_j - z'_j + \sum_{i:j\neq l(i)} (\lambda_i - \lambda'_i) \frac{\tau}{n} - \sum_{i:j=l(i)} (\lambda_i - \lambda'_i)\tau \end{aligned}$$

Since τ is finite, appropriately selecting λ'_i given λ_i for each i we can make $|f_j(z) - f_j(z')|$ sufficiently small corresponding to the value of $|z_j - z'_j|$ for each j, and so make $|f(z) - f(z')|$ sufficiently small corresponding to the value of $|z - z'|$. Thus, f is uniformly continuous, and then by the approximate version of Brouwer's fixed point theorem there exists a point z^* such that

$$|z^* - f(z^*)| < \varepsilon$$

for any $\varepsilon > 0$. Then, we obtain

$$|z^*_i - f_i(z^*)| < \varepsilon \text{ for all } i.$$

Let $\gamma > 0$ and $\tilde{z}$ be a point in $V(z^*, \gamma)$, where $V(z^*, \gamma)$ is a γ-neighborhood of z^*. If γ is sufficiently small, uniform continuity of f means

$$|\tilde{z}_i - f_i(\tilde{z})| < \varepsilon \tag{2}$$

for any $\varepsilon > 0$ and for all i. $\tilde{z}_i$ is the i-th component of $\tilde{z}$. Let Δ^* be a simplex of K which contains $\tilde{z}$.

> We cannot constructively determine which small simplex in K contains z^*. But we can constructively determine which small simplex has an intersection with $V(z^*, \gamma)$, or we may consider that if γ is sufficiently small, all points in $V(z^*, \gamma)$ are approximate fixed points.

Let $z^0, z^1, \dots, z^n$ be the vertices of Δ^*. Then, $\tilde{z}$ and $f(\tilde{z})$ are represented as

$$\tilde{z} = \sum_{i=0}^{n} \lambda_i z^i \text{ and } f(\tilde{z}) = \sum_{i=0}^{n} \lambda_i f(z^i),\ \lambda_i \geq 0,\ \sum_{i=0}^{n} \lambda_i = 1.$$

(1) implies that if only one z^k among $z^0, z^1, \dots, z^n$ is labeled with i, we have

$$|f_i(\tilde{z}) - \tilde{z}_i| = \left| \sum_{j=0}^{n} \lambda_j z_i^j + \sum_{j=0, j\neq k}^{n} \lambda_j \frac{\tau}{n} - \lambda_k \tau - z_i^* \right| = \left| \left(\frac{1}{n} \sum_{j=0, j\neq k}^{n} \lambda_j - \lambda_k \right) \tau \right| < \varepsilon.$$

Since ε may be arbitrarily small and $\tau > 0$, this means

$$\frac{1}{n} \sum_{j=0, j\neq k}^{n} \lambda_j - \lambda_k \approx 0.$$

(2) is satisfied with $\lambda_k \approx \frac{1}{n+1}$ for all k. On the other hand if no z^j is labeled with i, we have

$$f_i(\tilde{z}) = \sum_{j=0}^{n} \lambda_j z_i^j = \tilde{z}_i + (1 + \frac{1}{n})\tau,$$

and then (2) can not be satisfied. Thus, for each i one and only one z^j must be labeled with i. Therefore, Δ^* must be a fully labeled simplex.

We have completed the proof of Sperner's lemma by the approximate version of Brouwer's fixed point theorem.

Sperner's lemma alone is not sufficient to prove Brouwer's fixed point theorem. We need some non-constructive arguments. But Sperner's lemma is sufficient to constructively prove an approximate version of Brouwer's fixed point theorem. And conversely an approximate version of Brouwer's fixed point theorem is sufficient to prove Sperner's lemma.

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