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An Error Estimate For Thin Plate Spline Reconstruction On A Triangular Mesh*

Terhemen Aboiyar†

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Abstract

We present an error estimate for interpolation from cell averages with thin plate spline on structured triangular meshes. This situation arises in the construction of finite volume methods. We show that global reconstruction with thin plate splines yields first order approximation.

1 Introduction

Radial Basis Functions (RBFs) have become well-known as traditional and powerful tools for the multivariate interpolation of scattered data. Thin plate splines are a type of radial basis function that is frequently utilized in the literature for interpolation, for example, see Powell [5]. In recent years, radial basis functions have been utilized extensively in the numerical solution of partial differential equations like in Franke [3], and Behrens & Iske [2]. They have also been used in the recovery step of finite volume methods (Aboiyar [1]).

Traditionally, RBFs are used for the interpolation of scattered data. To this end, suppose d is a positive integer, Ω the closure of a bounded open set in $\mathbb{R}^d$ and $\mathbf{u} = (u(\mathbf{x}_1), \ldots, u(\mathbf{x}_n))^T \in \mathbb{R}^n$ a vector of function values sampled from an unknown function $u : \mathbb{R}^d \mapsto \mathbb{R}$ at a scattered finite point set $X = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\} \subseteq \Omega \subseteq \mathbb{R}^d$, $d \geq 1$, interpolation with radial basis functions requires the computing of a suitable interpolant $s : \mathbb{R}^d \mapsto \mathbb{R}$ satisfying

$$s(\mathbf{x}_j) = u(\mathbf{x}_j) \qquad \text{for all} \qquad j = 1, \ldots, n.$$

The RBF interpolation scheme utilizes a fixed radial function $\phi : [0, \infty) \mapsto \mathbb{R}$, and the interpolant s is taken to have the form

$$s(\mathbf{x}) = \sum_{j=1}^{n} c_j \phi(\|\mathbf{x} - \mathbf{x}_j\|) + p(\mathbf{x}), \qquad p \in \mathcal{P}_m^d$$

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†Department of Mathematics/Statistics/Computer Science, University of Agriculture, PMB 2373, Makurdi, Nigeria

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$. In addition, $\mathcal{P}_m^d$ denotes the vector space containing all real valued polynomials in d variables of degree at most $m-1$, where $m = m(\phi)$ is regarded to be the order of the basis function ϕ. Examples of popular RBFs can be found in [7].

Polyharmonic splines are a class of radial basis functions where

$$\phi \equiv \phi_{d,k}(r) = \begin{cases} r^{2k-d}\log(r), & \text{for } d \text{ even;} \\ r^{2k-d}, & \text{for } d \text{ odd,} \end{cases} \tag{1}$$

where k is required to satisfy $2k > d$ and the order is $m = k$. This class of RBFs includes the thin plate splines, where $\phi_{2,2}(r) = r^2 \log(r)$ and $m = 2$. In this case, the interpolant s has the form

$$s(\mathbf{x}) = \sum_{j=1} c_j \left(\|\mathbf{x} - \mathbf{x}_j\|^2 \log(\|\mathbf{x} - \mathbf{x}_j\|)\right) + d_1 + d_2 x_1 + d_3 x_2,$$

where we let x_1 and x_2 denote the two coordinates of $\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$.

The polyharmonic spline interpolation is optimal in its associated native space, the *Beppo Levi space* of order k defined as

$$\mathrm{BL}_k(\mathbb{R}^d) = \{u : D^\alpha u \in L^2(\mathbb{R}^d) \text{ for all } |\alpha| = k\} \subset C(\mathbb{R}^d),$$

being equipped with the semi-norm

$$|u|_{\mathrm{BL}_k(\mathbb{R}^d)} = \sum_{|\alpha|=k} \begin{pmatrix} k \\ \alpha \end{pmatrix} \|D^\alpha u\|^2_{L^2(\mathbb{R}^d)}.$$

Thin plate splines have been used in the reconstruction step of finite volume methods and error estimates were provided for interpolation for reconstruction on structured Cartesian grids by Gutzmer [4]. In this paper we extend his work, which is based on the earlier paper of Powell [5] for scattered data, to reconstruction on triangular meshes.

2 Generalized Interpolation

In certain applications, we may need to recover a function from other types of data associated with the function rather than point evaluations. The RBF interpolation algorithm can be extended to several other more general observation functionals. Following Wendland [7], let $\mathcal{H}$ be a Hilbert space and denote its dual by $\mathcal{H}'$. If $\Xi = \{\lambda_1, \ldots, \lambda_n\} \subseteq \mathcal{H}'$ is a set of linearly independent functionals on $\mathcal{H}$ and $u_1, \ldots, u_n \in \mathbb{R}$ are certain given values associated with u, then a generalized interpolation problem seeks to find a function $s \in \mathcal{H}$ such that

$$\lambda_i(s) = \lambda_i(u),\ i = 1, \ldots, n \quad \text{where} \quad \lambda_i(u) = u_i,\ i = 1, \ldots, n.$$

The interpolant s is referred to as the *generalized interpolant* and the generalized RBF interpolant has the form

$$s(\mathbf{x}) = \sum_{j=1}^{n} c_j \lambda_j^y \phi(\|\mathbf{x} - \mathbf{y}\|) + p(\mathbf{x}), \qquad \mathbf{x} \in \mathbb{R}^d \quad \text{and} \quad p \in \mathcal{P}_m^d$$

where the notation λ_j^y indicates the action of the functional λ_j on ϕ viewed as a function of the argument $\mathbf{y}$. We require this interpolant to satisfy

$$\lambda_i^x(s) = \lambda_i^x(u), \qquad i = 1, \ldots, n, \tag{2}$$

where λ_i^x indicates the action of the functional λ_i on s and u which are treated as functions of $\mathbf{x}$. To eliminate any additional degrees of freedom, the additional constraints

$$\sum_{j=1}^{n} c_j \lambda_j^x(p) = 0 \quad \text{for all} \quad p \in \mathcal{P}_m^d,$$

need to be satisfied.

We now turn to the case where the linearly independent functionals in Ξ are cell average operators. This situation arises in the recovery step of finite volume methods where point values of the unknown solution of a partial differential equation have to be reconstructed from cell average data, e.g. Sonar [6], Aboiyar *et al* [1].

3 Global Approximation

In this section, we will generalize the results of Powell [5] and Gutzmer [4] to instances where the interpolation data are given by cell averages on a triangular mesh instead of a Cartesian grid or at scattered point values.

If we divide a region $\Omega \in \mathbb{R}^2$ into non-overlapping subregions $\mathcal{T} = \{V_j\}$, then for some integrable function u, the *cell average operators* are defined as

$$\lambda_j^x(u) := \bar{u}_j = \frac{1}{|V_j|} \int_{V_j} u(\mathbf{x})\, d\mathbf{x}.$$

We will focus on a pointwise error estimate of thin plate spline reconstruction on triangular meshes. Based on the earlier work of Powell [5] and Gutzmer [4], we present a pointwise error estimate for thin plate spline interpolation for situations where interpolation data are cell averages on a triangular mesh. In Powell [5], the results were provided for interpolation of scattered point values while Gutzmer [4] treated the instance where the interpolation data were cell averages on Cartesian grids.

Let $u : \mathbb{R}^2 \to \mathbb{R}$ be an integrable function. Then the thin plate spline interpolant s subject to the conditions $\lambda_i^x(s) = \lambda_i^x(u)$, $i = 1, \ldots, n$, has the form

$$s(\mathbf{x}) = \sum_{i=1}^{n} c_i \lambda_i^y \left(\|\mathbf{x} - \mathbf{y}\|^2 \log(\|\mathbf{x} - \mathbf{y}\|) \right) + d_1 + d_2 x_1 + d_3 x_2, \tag{3}$$

where $\mathbf{x} = (x_1, x_2)^T$ and $\mathbf{y} = (y_1, y_2)^T$.

We first of all state the following lemma.

LEMMA 1 (Powell [5],Gutzmer [4]). Let λ_i^x, $i = 0, \ldots, n$ be a set of $n > 3$ functionals with compact support and unisolvent on $\mathcal{P}_2^2$. If

$$\sum_{i=0}^{n} \hat{\beta}_i = 0 \quad \text{and} \quad \sum_{i=0}^{n} \hat{\beta}_i \lambda_i^x(p) = 0 \quad \text{for all} \quad p \in \mathcal{P}_2^2, \tag{4}$$

then the functional $\hat{L} = \sum_{i=0}^{n} \hat{\beta}_i \lambda_i^x$ can be bounded as follows

$$\left|\hat{L}g\right| \le \left[8\pi \|g\|_{\mathrm{BL}_2}^2 \sum_{i=0}^{n}\sum_{j=0}^{n} \hat{\beta}_i\hat{\beta}_j \lambda_i^x \lambda_j^y \phi_{2,2}(\|\mathbf{x}-\mathbf{y}\|)\right]^{1/2}, \tag{5}$$

for any $g \in \mathrm{BL}_2(\mathbb{R}^2)$, $\mathbf{x} = (x_1, x_2)^T$, $\mathbf{y} = (y_1, y_2)^T$ and $\phi_{2,2}(r) = r^2 \log(r)$, $r \ge 0$.

This lemma enables us to estimate the error at a given point $\tilde{\mathbf{x}}$, if the interpolation data are cell averages.

We now prove a key result that enables us to obtain error estimates for unstructured triangular meshes.

THEOREM 1. Let the triangles T_i, $i = 1, \ldots, n$ with vertices $\mathbf{a}_{i1}, \mathbf{a}_{i2}, \mathbf{a}_{i3}$ and centres $\mathbf{a}_{ic} = (\mathbf{a}_{i1} + \mathbf{a}_{i2} + \mathbf{a}_{i3})/3$ be assigned to the functionals (cell average operators) λ_i^x, $i = 1, \ldots, n$ defined by

$$\lambda_i^x(u) := \frac{1}{|T_i|}\int_{T_i} u(\mathbf{x})\, d\mathbf{x}, \qquad i = 1, \ldots, n.$$

Let $\lambda_0^x = \delta_{\tilde{\mathbf{x}}}$ be the point evaluation at $\tilde{\mathbf{x}}$ and let $\hat{\beta}_i$, $i = 1, \ldots, n$ be given by

$$\hat{\beta}_0 = -1, \tag{6}$$

$$\hat{\beta}_i = \beta_i, \quad \beta_i > 0, \qquad i = 1, \ldots, n, \text{ and } \quad \sum_{i=1}^{n} \beta_i = 1, \tag{7}$$

such that

$$\tilde{\mathbf{x}} = \sum_{i=1}^{n} \beta_i \mathbf{a}_{ic}.$$

Then we obtain

$$|u(\tilde{\mathbf{x}}) - s(\tilde{\mathbf{x}})| \le \left[8\pi \|u\|_{\mathrm{BL}_2}^2 \Phi(\beta)\right]^{1/2} \tag{8}$$

for all $u \in \mathrm{BL}_2$, where $\beta = \{\beta_i\}_{i=1}^n$ and Φ is given by

$$\Phi(\beta) = \sum_{i=1}^{n}\sum_{j=1}^{n} \beta_i\beta_j \lambda_i^x \lambda_j^y \phi_{2,2}(\|\mathbf{x}-\mathbf{y}\|) - 2\sum_{i=1}^{n} \beta_i \lambda_i^y \phi_{2,2}(\|\tilde{\mathbf{x}}-\mathbf{y}\|), \tag{9}$$

and s denotes the thin plate spline interpolant with respect to the data $\lambda_i^x(u) = \lambda_i^x(s)$, $i = 1, \ldots, n$.

PROOF. Let $g = u - s$ so that

$$\hat{L}g = \sum_{i=0}^{n} \hat{\beta}_i \lambda_i^x g = s(\tilde{\mathbf{x}}) - u(\tilde{\mathbf{x}}).$$

To be able to use the result (5) in Lemma 1 in the proof of this theorem, we need to make sure that the two conditions on the $\hat{\beta}_i$'s in (4) are satisfied. Clearly, with our choices of $\hat{\beta}_i$, $i = 0, 1, \ldots, n$ in (6) and (7), the first condition is satisfied.

To show that the second condition is satisfied, we need to evaluate

$$\sum_{i=0}^{n} \hat{\beta}_i \lambda_i^x \mathbf{x} = \sum_{i=0}^{n} \hat{\beta}_i \lambda_i^x \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We do this by mapping each triangle T_i with vertices $\mathbf{a}_{i1} = (x_{1i}^1, x_{2i}^1)$, $\mathbf{a}_{i2} = (x_{1i}^2, x_{2i}^2)$, $\mathbf{a}_{i3} = (x_{1i}^3, x_{2i}^3)$ to a canonical reference triangle K with vertices $\hat{\mathbf{a}}_1 = (0,0)$, $\hat{\mathbf{a}}_2 = (1,0)$, $\hat{\mathbf{a}}_3 = (0,1)$ by a unique invertible affine mapping F_i such that

$$\mathbf{x} = F_i(\mathbf{v}) = B_i \mathbf{v} + \mathbf{a}_{i1}, \tag{10}$$

where $\mathbf{x} = (x_1, x_2) \in T_i$, $\mathbf{v} = (v_1, v_2) \in \mathrm{K}$, B_i is an invertible 2×2 matrix and

$$F_i(\hat{\mathbf{a}}_\ell) = \mathbf{a}_{i\ell}, \qquad \ell = 1, 2, 3.$$

The matrix B_i is given as

$$B_i = \begin{pmatrix} x_{1i}^2 - x_{1i}^1 & x_{1i}^3 - x_{1i}^1 \\ x_{2i}^2 - x_{2i}^1 & x_{2i}^3 - x_{2i}^1 \end{pmatrix}. \tag{11}$$

Hence, we have the relations

$$\begin{aligned} x_1 &= x_{1i}^1 + (x_{1i}^2 - x_{1i}^1) v_1 + (x_{1i}^3 - x_{1i}^1) v_2, \\ x_2 &= x_{2i}^1 + (x_{2i}^2 - x_{2i}^1) v_1 + (x_{2i}^3 - x_{2i}^1) v_2. \end{aligned}$$

If we invert this relationship, we find that

$$\begin{aligned} v_1 &= \frac{(x_1 - x_{1i}^1)(x_{2i}^3 - x_{2i}^1) - (x_2 - x_{2i}^1)(x_{1i}^3 - x_{1i}^1)}{J_i} \\ v_2 &= \frac{(x_2 - x_{2i}^1)(x_{1i}^2 - x_{1i}^1) - (x_1 - x_{1i}^1)(x_{2i}^2 - x_{2i}^1)}{J_i}, \end{aligned}$$

where the Jacobian J_i of the mapping is given by

$$J_i = \det(B_i).$$

Now, $|T_i| = J_i |\mathrm{K}|$, $|\mathrm{K}| = \frac{1}{2}$ and $dx_1 dx_2 = J_i\, dv_1 dv_2$; therefore,

$$\int_{T_i} x_1\, dx_1\, dx_2 = \frac{1}{6} J_i \left(x_{1i}^1 + x_{1i}^2 + x_{1i}^3 \right) \quad \text{and} \quad \int_{T_i} x_2\, dx_1\, dx_2 = \frac{1}{6} J_i \left(x_{2i}^1 + x_{2i}^2 + x_{2i}^3 \right).$$

All this means that

$$\begin{aligned} \sum_{i=0}^{n} \hat{\beta}_i \lambda_i^x \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= -\widetilde{\mathbf{x}} + \sum_{i=1}^{n} \frac{\beta_i}{|T_i|} \begin{pmatrix} \frac{1}{6} J_i (x_{1i}^1 + x_{1i}^2 + x_{1i}^3) \\ \frac{1}{6} J_i (x_{2i}^1 + x_{2i}^2 + x_{2i}^3) \end{pmatrix} \\ &= -\widetilde{\mathbf{x}} + \sum_{i=1}^{n} \frac{\beta_i}{J_i |K|} \frac{1}{2} J_i \mathbf{a}_{ic} \\ &= -\widetilde{\mathbf{x}} + \sum_{i=1}^{n} \beta_i \mathbf{a}_{ic} \\ &= 0, \end{aligned} \tag{12}$$

showing that the second condition is also satisfied. We then conclude by Lemma 1 that

$$|u(\tilde{\mathbf{x}}) - s(\tilde{\mathbf{x}})| \leq \left[8\pi \|g\|^2_{\mathrm{BL}_2} \Phi(\beta)\right]^{1/2}. \tag{13}$$

Since the interpolant s minimizes the energy $|\cdot|_{\mathrm{BL}_2}$ among all interpolants $f \in \mathrm{BL}_2$ satisfying

$$\lambda_i^x f = \lambda_i^x u, \qquad i = 1, \dots, n$$

we obtain

$$\begin{aligned}
\|g\|^2_{\mathrm{BL}_2} &= \|u - s\|^2_{\mathrm{BL}_2} = (u - s, u - s)_{\mathrm{BL}_2} \\
&= (u, u)_{\mathrm{BL}_2} - (u, s)_{\mathrm{BL}_2} - 2(s, u - s)_{\mathrm{BL}_2} + (s, u - s)_{\mathrm{BL}_2} \\
&= (u, u)_{\mathrm{BL}_2} - 2(s, u - s)_{\mathrm{BL}_2} - (s, s)_{\mathrm{BL}_2} \\
&= \|u\|^2_{\mathrm{BL}_2} - 2\underbrace{(s, u - s)_{\mathrm{BL}_2}}_{=0} - \|s\|^2_{\mathrm{BL}_2} \\
&\leq \|u\|^2_{\mathrm{BL}_2}.
\end{aligned} \tag{14}$$

This concludes the proof.

A more precise form of the error bound (13) can be obtained by finding an estimate of the quadratic form $\Phi(\beta)$

4 Example: An Estimate for $\Phi(\beta)$

We estimate $\Phi(\beta)$ by considering the special case where our mesh contains two triangles T_i and T_j with a common edge that has center $\tilde{x}$. We can map T_i to a triangle K_i with vertices $(0,0)$,$(h,0)$ and $(0,h)$ and T_j to a triangle K_j with vertices $(0,h)$,$(h,0)$ and (h,h) with the further condition that $\tilde{x} \mapsto \bar{x} = (h/2, h/2)$. This is a uniform grid of right angled triangles with common edge at centre $\tilde{x}$. We take all the coefficients β to be zero except for T_i and T_j.

We can write $\Phi(\beta)$, where we take $\phi \equiv \phi_{2,2}$, as:

$$\begin{aligned}
\Phi(\beta) = \; & \beta_i\beta_i\lambda_i^x\lambda_j^y\phi(\|x-y\|) + 2\beta_i\beta_j\lambda_i^x\lambda_j^y\phi(\|x-y\|) + \beta_j\beta_j\lambda_j^x\lambda_j^y\phi(\|x-y\|) \\
& -2\beta_i\lambda_i^y\phi(\|\tilde{x}-y\|) - 2\beta_j\lambda_y^j\phi(\|\tilde{x}-y\|)
\end{aligned}$$

If we set $\beta_i = \frac{1}{2} = \beta_j$ and since $|K_i| = |K_j| = \frac{1}{2}h^2$, we can rewrite Φ as

$$\begin{aligned}
\Phi(\beta) &= \frac{1}{2}\lambda_i^x\lambda_j^y\phi(\|x-y\|) + \frac{1}{4}\lambda_i^x\lambda_j^y\phi(\|x-y\|) + \frac{1}{4}\lambda_j^x\lambda_j^y\phi(\|x-y\|) \\
&\quad -\lambda_i^y\phi(\|\tilde{x}-y\|) - \lambda_j^y\phi(\|\tilde{x}-y\|) \\
&= \frac{2}{h^4}\int_{K_i}\int_{K_j}\phi(\|x-y\|) + \frac{1}{h^4}\int_{K_i}\int_{K_i}\phi(\|x-y\|) + \frac{1}{h^4}\int_{K_j}\int_{K_j}\phi(\|x-y\|) \\
&\quad -\frac{2}{h^2}\int_{K_i}\phi(\|\tilde{x}-y\|) - \frac{2}{h^2}\int_{K_j}\phi(\|\tilde{x}-y\|)
\end{aligned} \tag{15}$$

Now,

$\phi(\|x-y\|) = \|x-y\|^2 \log \|x-y\| = [(x_1-y_1)^2+(x_2-y_2)^2] \log[(x_1-y_1)^2+(x_2-y_2)^2]^{1/2}$.

Let $v_i = (0,0), v_j = (h,0), v_k = (0,h), v_l = (h,h)$ and define $\hat{\xi} = \xi/h, \hat{\eta} = \eta/h$ so that $\hat{v}_i = (0,0), \hat{v}_j = (1,0), \hat{v}_k = (0,1), \hat{v}_l = (1,1)$.Then, the triangle $\hat{K}_i$ has vertices $(0,0)$, $(1,0)$ and $(0,1)$ and triangle $\hat{K}_j$ has vertices $(0,1)$, $(1,0)$ and $(1,1)$. A simple substitution gives

$$\|x-y\|^2 \log \|x-y\| = h^2 \phi(\|\hat{x}-\hat{y}\|) + h^2 \log(h)[(\hat{x}_1-\hat{y}_1)^2 + (\hat{x}_2-\hat{y}_2)^2]$$

where $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $dy_1 dy_2 dx_1 dx_2 = h^4 d\hat{y}_1 d\hat{y}_2 d\hat{x}_1 d\hat{x}_2$. Let $D = [(\hat{x}_1-\hat{y}_1)^2 + (\hat{x}_2-\hat{y}_2)^2]$, $d\hat{y}_1 d\hat{y}_2 d\hat{x}_1 d\hat{x}_2 = dX$ and dropping the hats in subsequent calculations give

$$\begin{aligned}
\Phi(\beta) &= \frac{2}{h^4}\int_0^1\int_0^{1-x_1}\int_0^1\int_{1-y_1}^1 (h^2\phi(\|x-y\|) + h^2\log(h)D)h^4 dX \\
&\quad + \frac{1}{h^4}\int_0^1\int_0^{1-x_1}\int_0^1\int_0^{1-x_1} (h^2\phi(\|x-y\|) + h^2\log(h)D)h^4 dX \\
&\quad + \frac{1}{h^4}\int_0^1\int_{1-y_1}^1\int_0^1\int_{1-y_1}^1 (h^2\phi(\|x-y\|) + h^2\log(h)D)h^4 dX \\
&\quad - \frac{2}{h^2}\int_0^1\int_0^{1-x_1} (h^2\phi(\|\tilde{x}-y\|) + h^2\log(h)\tilde{D})h^2 dy_2 dy_1 \\
&\quad - \frac{2}{h^2}\int_0^1\int_{1-y_1}^1 (h^2\phi(\|\tilde{x}-y\|) + h^2\log(h)\tilde{D})h^2 dy_2 dy_1 \\
&= h^2\int_0^1\int_0^1\int_0^1\int_0^1 \phi(\|x-y\|)dX + 2h^2\log(h)\int_0^1\int_0^1\int_0^1\int_0^1 D dX \\
&\quad - 2h^2\int_0^1\int_0^1 \phi(\|\tilde{x}-y\|)dy_2 dy_1 - 2h^2\log(h)\int_0^1\int_0^1 \tilde{D} dy_2 dy_1 \\
&= h^2[c_1 - 2c_2] + h^2\log(h)\left\{\int_0^1\int_0^1\int_0^1\int_0^1 D dX - \int_0^1\int_0^1 2\tilde{D} dy_2 dy_1\right\} \\
&= Ch^2 + h^2\log(h)\left\{\int_0^1\int_0^1\int_0^1\int_0^1 D dX - \int_0^1\int_0^1 2\tilde{D} dy_2 dy_1\right\}
\end{aligned}$$

Since $\tilde{x} = (\frac{1}{2}, \frac{1}{2})$, we have

$$\tilde{D} = (\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2 = \left(\frac{1}{2} - y_1\right)^2 + \left(\frac{1}{2} - y_2\right)^2$$

and

$$D = (x_1 - y_1)^2 + (x_2 - y_2)^2.$$

Using MAPLE we obtain

$$\int_0^1\int_0^1\int_0^1\int_0^1 D dX = \frac{1}{3} \qquad \text{and} \qquad \int_0^1\int_0^1 \tilde{D} dy_2 dy_1 = \frac{1}{6}.$$

Therefore,

$$\Phi(\beta) = Ch^2 + h^2 \log(h) \underbrace{\left(\frac{1}{3} - 2 \cdot \frac{1}{6}\right)}_{=0} = Ch^2. \tag{16}$$

All this leads to the following result:

THEOREM 2. Given a regular mesh of containing 2 triangles of size h and if x is the midpoint of the interior edge. We obtain first order accuracy of the thin plate spline reconstruction,

$$|u(x) - s_u(x)| \leq C_t \|u\| h,$$

where $C_t = (8\pi C)^{-1/2}$ for all $u \in \mathrm{BL}_2$, and s_u denotes the thin plate spline reconstruction of u.

PROOF. This is obtained by placing (16) in (8)

5 Conclusion

We have shown in this paper that the approximation order for thin plate spline interpolation on a triangular mesh is one.

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References

[1] T. Aboiyar, E. H. Georgoulis and A. Iske, High order WENO finite volume schemes using polyharmonic spline recontruction, Proceedings of the International Conference on Numerical Analysis and Approximation Theory, Cluj-Napoca, pp. 113-126, 2006.

[2] J. Behrens and A. Iske, Grid-free adaptive semi-Lagrangian advection using radial basis functions, Comp. Math. Appl., 43(2002), 319–327.

[3] C. Franke, Solving partial differential equations by collocation using radial basis functions, Appl. Math. Comput. 93(1998), 73–82.

[4] T. Gutzmer, Error estimates for reconstruction using thin plate spline interpolants. Technical Report 97-08, ETH, 1997.

[5] M. J. D. Powell, The uniform convergence of thin plate spline interpolation in two dimensions, Numer. Math., 68(1993), 107–128.

[6] T. Sonar, Optimal recovery using thin plate splines in finite volume methods for the numerical solution of hyperbolic conservation laws. IMA Journal of Numerical Analysis, 16(1996), 549–581.

[7] H. Wendland, Scattered Data Approximation, Cambridge University Press, 2005.