Applied Mathematics E-Notes, 11(2011), 208–215 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Positive Periodic Solutions For A Class Of Higher Order Functional Difference Equations*

Xinhong Chen and Weibing Wang†

Received 6 September 2010

Abstract

In this paper, we apply a fixed point theorem to obtain sufficient conditions for the existence, multiplicity and nonexistence of positive ω-periodic solutions for a class of higher-order functional difference equations.

1 Introduction

In this paper, we investigate the existence, multiplicity and nonexistence of positive ω-periodic solutions for the periodic equation.

$$x(n+m) = g(x(n))x(n) - \lambda b(n)f(x(n-\tau(n))), \tag{1}$$

where $\lambda > 0$ is a positive parameter and we make the assumptions:

(H_1) $b, \tau : Z \to Z$ are ω-periodic sequences, $b(n) > 0$, $\omega, m \in N, (m, \omega) = 1$, here (m, ω) is the greatest common divisor of m and ω.

(H_2) $f, g : [0, +\infty) \to [0, +\infty)$ are continuous. $1 < l < g(u) < L < +\infty$ for $u \geq 0, f(u) > 0$ for $u > 0$.

The existence of positive periodic solutions of discrete mathematical models has been studied extensively in recent years, see [1, 2, 5, 6, 7, 9, 10, 11] and the references therein. For example, Raffoul [3] considered the existence of positive periodic solutions for functional difference equations with parameter

$$x(n+1) = a(n)x(n) + \lambda h(n)g(x(n-\tau(n))). \tag{2}$$

Jiang [4] obtained the optimal existence theorem for single and multiple positive periodic solutions to general functional difference equations

$$\Delta x(n) = -a(n)x(n) + g(n, x(n-\tau(n))). \tag{3}$$

However, relatively few paper has discussed existence of positive periodic solutions for higher-order functional difference equations. In this paper, we apply a fixed point theorem to discuss the existence, multiple and nonexistence of positive ω-periodic solutions of (1).

*Mathematics Subject Classifications: 38A12

†Department of Mathematics, Hunan University of Science and Technology, Xiangtan, Hunan 411201, P.R. China

2 Preliminaries

Let $X = \{x : Z \to R, x(n+\omega) = x(n)\}$. When endowed with the maximum norm $\|x\| = \max_{n\in[0,\omega]} |x(n)|$, X is a Banach space. From (1), we have that for any $x \in X$,

$$\frac{1}{g(x(n))}x(n+m) - x(n) = -\frac{\lambda b(n)}{g(x(n))}f(x(n-\tau(n))),$$

$$\begin{aligned} & \frac{1}{g(x(n))g(x(n+m))}x(n+2m) - \frac{1}{g(x(n))}x(n+m) \\ = & -\frac{\lambda b(n+m)}{g(x(n))g(x(n+m))}f(x(n+m-\tau(n+m))), \end{aligned}$$

$$\cdots\cdots$$

$$\begin{aligned} & (\prod_{i=0}^{\omega-1}\frac{1}{g(x(n+im))})x(n+\omega m) - (\prod_{i=0}^{\omega-2}\frac{1}{g(x(n+im))})x(n+(\omega-1)m) \\ = & -\lambda(\prod_{i=0}^{\omega-1}\frac{1}{g(x(n+im))})b(n+(\omega-1)m)f(x(n+(\omega-1)m-\tau(n+(\omega-1)m))). \end{aligned}$$

By summing the above equations and using periodicity of x, we obtain

$$x(n) = \sum_{i=0}^{\omega-1}\frac{\prod_{j=0}^{i}\frac{1}{g(x(n+jm))}}{1-\prod_{t=0}^{\omega-1}\frac{1}{g(x(n+tm))}}\lambda b(n+im)f(x(n+im-\tau(n+im))). \quad (4)$$

Define the map $T_\lambda : X \to X$ and a cone P in X by

$$T_\lambda x(n) = \lambda\sum_{i=0}^{\omega-1}G(n,i)b(n+im)f(x(n+im-\tau(n+im))),$$

$$P = \{x \in X : x(n) \geq \delta\|x\|, n \in [0,\omega]\},$$

respectively, where

$$G(n,i) = \left(\prod_{j=0}^{i}\frac{1}{g(x(n+jm))}\right)\left(1-\prod_{t=0}^{\omega-1}\frac{1}{g(x(n+tm))}\right)^{-1},$$

and

$$\delta = \frac{l^\omega - 1}{L^\omega - 1}.$$

Clearly, $\delta \in (0,1)$ and

$$\frac{1}{L^\omega - 1} \leq G(n,i) \leq \frac{1}{l^\omega - 1}, 0 \leq i \leq \omega - 1.$$

Further, one can easily show that the fixed point of T_λ in P is the positive periodic solution of (1). The following well-known result of the fixed point theorem is crucial in our arguments.

LEMMA 2.1 ([8]). Let E be a Banach space and P be a cone in E. Suppose Λ_1 and Λ_2 are open subsets of E such that $0 \in \Lambda_1 \subset \bar{\Lambda}_1 \subset \Lambda_2$ and suppose that

$$T_\lambda : P \cap (\bar{\Lambda}_2 \backslash \Lambda_1) \to P$$

is a completely continuous operator. If one of the following conditions is satisfied,

(i) $\|T_\lambda x\| \le \|x\|$ for $x \in P \cap \partial\Lambda_1$; $\|T_\lambda x\| \ge \|x\|$ for $x \in P \cap \partial\Lambda_2$;

(ii) $\|T_\lambda x\| \ge \|x\|$ for $x \in P \cap \partial\Lambda_1$; $\|T_\lambda x\| \le \|x\|$ for $x \in P \cap \partial\Lambda_2$.

Then T_λ has a fixed point in $P \cap (\bar{\Lambda}_2 \backslash \Lambda_1)$.

LEMMA 2.2. Assume (H_1)-(H_2) hold. Then $T_\lambda(P) \subset P$ and $T_\lambda : P \to P$ is completely continuous.

PROOF. In view of the definition of P, for $x \in P$, we have

$$\begin{aligned} T_\lambda x(n+\omega) &= \lambda \sum_{i=0}^{\omega-1} G(n+\omega, i) b(n+\omega+im) f(x(n+\omega+im-\tau(n+\omega+im))) \\ &= \lambda \sum_{i=0}^{\omega-1} G(n,i) b(n+im) f(x(n+im-\tau(n+im))) = T_\lambda x(n). \end{aligned}$$

On the other hand,

$$\begin{aligned} T_\lambda x(n) &\ge \frac{1}{L^\omega - 1} \lambda \sum_{i=0}^{\omega-1} b(n+im) f(x(n+im-\tau(n+im))) \\ &= \frac{1}{L^\omega - 1} \lambda \sum_{j=0}^{\omega-1} b(jm) f(x(jm-\tau(jm))) \\ &= \frac{1}{L^\omega - 1} \lambda \sum_{j=0}^{\omega-1} b(j) f(x(j-\tau(j))), \end{aligned}$$

and

$$T_\lambda x(n) \le \frac{1}{l^\omega - 1} \lambda \sum_{j=0}^{\omega-1} b(j) f(x(j-\tau(j))).$$

Hence,

$$T_\lambda x(n) \ge \frac{l^\omega - 1}{L^\omega - 1} \|T_\lambda x\| = \delta \|T_\lambda x\|.$$

Thus $T_\lambda(P) \subset P$ and according to Arzela-Ascoli's Theorem, it is easy to show that $T_\lambda : P \to P$ is completely continuous. The proof is complete.

LEMMA 2.3. Assume (H_1)-(H_2) hold and let $\varepsilon > 0$. If $f(u) \geq u\varepsilon$ for any $u > 0$, then for any $x \in P$,

$$\|T_\lambda x\| \geq \lambda\varepsilon \frac{l^\omega - 1}{(L^\omega - 1)^2} \sum_{j=0}^{\omega-1} b(j)\|x\|.$$

PROOF. Since $x \in P$ and $f(u) \geq u\varepsilon$, we have

$$\begin{aligned}
T_\lambda x(n) &\geq \frac{1}{L^\omega - 1}\lambda \sum_{i=0}^{\omega-1} b(n+im) f(x(n+im-\tau(n+im))) \\
&\geq \frac{1}{L^\omega - 1}\lambda \sum_{i=0}^{\omega-1} b(n+im) x(n+im-\tau(n+im))\varepsilon \\
&\geq \frac{1}{L^\omega - 1}\lambda \sum_{j=0}^{\omega-1} b(j) \frac{l^\omega - 1}{L^\omega - 1}\|x\|\varepsilon \\
&= \frac{l^\omega - 1}{(L^\omega - 1)^2}\lambda \sum_{j=0}^{\omega-1} b(j)\|x\|\varepsilon.
\end{aligned}$$

Thus

$$\|T_\lambda x\| \geq \lambda\varepsilon \frac{l^\omega - 1}{(L^\omega - 1)^2} \sum_{j=0}^{\omega-1} b(j)\|x\|.$$

LEMMA 2.4. Assume (H_1)-(H_2) hold . If there exists an $\eta > 0$ such that $f(u) \leq \eta u$ for any $u > 0$, then for $x \in P$

$$\|T_\lambda x\| \leq \frac{1}{l^\omega - 1}\lambda\eta \sum_{j=0}^{\omega-1} b(j)\|x\|.$$

This Lemma can be shown in a similar manner as in Lemma 2.3.

3 Main Results

Let $\Omega_r = \{x \in P : \|x\| < r\}$. Then $\partial\Omega_r = \{x \in P : \|x\| = r\}$. Put

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}, \tag{5}$$

$I_0 =$ number of zeros in the set$\{f_0, f_\infty\}$, $I_\infty =$ number of infinitions in the set$\{f_0, f_\infty\}$,

$$m(r) = \min\{f(x) : x \in [\delta r, r], r > 0\}, M(r) = \max\{f(x) : x \in [\delta r, r], r > 0\}.$$

At first, we discuss the existence and multiplicity of positive periodic solutions for (1).

THEOREM 3.1. Assume (H_1)-(H_2) hold. If $I_0 = 1$ or 2, then (1) has at least one I_0 positive ω-periodic solution for $\lambda > \lambda_*$ where

$$\lambda_* = \frac{L^\omega - 1}{\sum_{j=0}^{\omega-1} b(j)} \inf_{r>0} \frac{r}{m(r)}.$$

PROOF. Choose $r_1 > 0$ such that

$$\inf_{r>0} \frac{r}{m(r)} < \frac{r_1}{m(r_1)} \leq \frac{\sum_{j=0}^{\omega-1} b(j)}{L^\omega - 1}\lambda.$$

Noting that $f(u) \geq m(r_1)$ for $u \in \partial\Omega_{r_1}$, we can easily get

$$\|T_\lambda x\| \geq \lambda \frac{\sum_{j=0}^{\omega-1} b(j)m(r_1)}{L^\omega - 1}, x \in \partial\Omega_{r_1}.$$

Hence,

$$\|T_\lambda x\| > \|x\|, \quad \text{for } x \in \partial\Omega_{r_1} \quad \text{and} \quad \lambda > \lambda_*.$$

If $f_0 = 0$, we choose $0 < r_2 < r_1$ such that $f(x) \leq \eta x$ for $0 \leq x \leq r_2$, where $\eta > 0$ satisfies

$$\lambda\eta \frac{1}{l^\omega - 1} \sum_{j=0}^{\omega-1} b(j) < 1.$$

According to Lemma 2.4, we have for $x \in \partial\Omega_{r_2}$,

$$\|T_\lambda x\| \leq \frac{1}{l^\omega - 1}\lambda\eta \sum_{j=0}^{\omega-1} b(j)\|x\| \leq \|x\|.$$

Then T_λ has a fixed point in $P \cap (\bar{\Omega}_{r_1} \setminus \Omega_{r_2})$, which is a positive ω-periodic solution of (1) for $\lambda > \lambda_*$.

If $f_\infty = 0$, there is a $K > 0$ such that $f(x) \leq \eta x$ for $x \geq K$, where $\eta > 0$ satisfies

$$\lambda\eta \frac{1}{l^\omega - 1} \sum_{j=0}^{\omega-1} b(j) < 1.$$

Let $r_3 = \max\{2r_1, \frac{K}{\delta}\}$, then $x(n) \geq \delta\|x\| \geq K$ for $x \in \partial\Omega_{r_3}$ and $n \in [0, \omega]$. Thus $f(x) \leq \eta x$ for $x \in \partial\Omega_{r_3}$. In view of Lemma 2.4, we have

$$\|T_\lambda x\| \leq \frac{1}{l^\omega - 1}\lambda\eta \sum_{j=0}^{\omega-1} b(j)\|x\| \leq \|x\|, \text{for } x \in \partial\Omega_{r_3}.$$

Then T_λ has a fixed point in $P \cap (\bar{\Omega}_{r_3} \setminus \Omega_{r_1})$, and (1) has at least one positive ω-periodic solution for $\lambda > \lambda_*$.

If $f_0 = f_\infty = 0$, it is easy to see from the above proof that T_λ has a fixed point x_1 in $\bar{\Omega}_{r_1} \setminus \Omega_{r_2}$ and a fixed point x_2 in $\bar{\Omega}_{r_3} \setminus \Omega_{r_1}$ such that

$$r_2 < \|x_1\| < r_1 < \|x_2\| < r_3.$$

Consequently, (1) has at least two positive ω-periodic solutions for $\lambda > \lambda_*$. The proof is complete.

Similar to that of the Theorem 3.1, we have

THEOREM 3.2. Assume (H_1)-(H_2) hold. If $I_\infty = 1$ or 2, then (1) has at least one I_∞ positive ω-periodic solution for $0 < \lambda < \frac{l^\omega - 1}{\sum_{j=0}^{\omega-1} b(j)} \sup_{r>0} \frac{r}{M(r)}$.

Next, we consider the nonexistence of positive ω-periodic solutions for (1).

THEOREM 3.3. Assume (H_1)-(H_2) hold. If $I_0 = 0$ (or $I_\infty = 0$), then (1) has no positive ω-periodic solutions for sufficiently large $\lambda > 0$ (or sufficiently small $\lambda > 0$).

PROOF. Since $I_0 = 0$, we have $f_0 > 0$ and $f_\infty > 0$, there exist $\varepsilon_1 > 0, \varepsilon_2 > 0$ and $\gamma_2 > \gamma_1 > 0$ such that

$$f(x) \geq \varepsilon_1 x \quad \text{for} \quad x \in [0, \gamma_1], \qquad f(x) \geq \varepsilon_2 x \text{ for } \quad x \in [\gamma_2, \infty).$$

Let

$$c_1 = \min \left\{ \varepsilon_1, \varepsilon_2, \min_{\gamma_1 \leq x \leq \gamma_2} \{ \frac{f(x)}{x} \} \right\}.$$

Then $f(x) \geq c_1 x$ for $x \in [0, \infty)$. Assume y is a positive ω-periodic solution of (1). We show that this leads to a contradiction for $\lambda > \bar{\lambda}$, where

$$\bar{\lambda} = \frac{(L^\omega - 1)^2}{(l^\omega - 1) c_1 \sum_{j=0}^{\omega-1} b(j)}.$$

Since $T_\lambda y = y$, it follows from Lemma 2.3 that for $\lambda > \bar{\lambda}$,

$$\|y\| = \|T_\lambda y\| \geq \lambda \frac{l^\omega - 1}{(L^\omega - 1)^2} c_1 \sum_{j=0}^{\omega-1} b(j) \|y\| > \|y\|,$$

which is a contradiction.

If $I_\infty = 0$, then $f_0 < \infty$ and$f_\infty < \infty$. There exists $\eta_1 > 0, \eta_2 > 0, \gamma_2 > \gamma_1 > 0$ such that

$$f(x) \leq \eta_1 x \text{ for } x \in [0, \gamma_1], \quad f(x) \leq \eta_2 x \quad \text{for} \quad x \in [\gamma_2, \infty).$$

Let $c_2 = \max\{\eta_1, \eta_2, \max\{\frac{f(x)}{x}\}\}$, we have

$$f(x) \leq c_2 x \quad \text{for } x \in [0, \infty).$$

Assume y is a positive ω-periodic solution of (1). We show that this leads to a contradiction for $0 < \lambda < \bar{\lambda}$, where

$$\bar{\lambda} = \frac{l^\omega - 1}{c_2 \sum_{j=0}^{\omega-1} b(j)}.$$

Since $T_\lambda y = y$, it follows from Lemma 2.4 that for $0 < \lambda < \bar{\lambda}$,

$$\|y\| = \|T_\lambda y\| \leq \lambda \frac{1}{l^\omega - 1} c_2 \sum_{j=0}^{\omega-1} b(j) \|y\| < \|y\|,$$

which is a contradiction. The proof is complete.

COROLLARY 3.1. Assume that (H_1)-(H_2) hold. If there is a $M_1 > 0$ such that $f(x) \geq M_1 x$ for $x \in [0, \infty)$, then there exists a $\bar{\lambda} = \frac{(L^\omega - 1)^2}{(l^\omega - 1)\sum_{j=0}^{\omega-1} b(j) M_1}$ such that for all $\lambda > \bar{\lambda}$, (1) has no positive ω-periodic solution.

COROLLARY 3.2. Assume that (H_1)-(H_2) hold. If there is a $M_2 > 0$ such that $f(x) \leq M_2 x$ for $x \in [0, \infty)$, then there exists a $\bar{\lambda} = \frac{l^\omega - 1}{\sum_{j=0}^{\omega-1} b(j) M_2}$ such that for all $0 < \lambda < \bar{\lambda}$, (1) has no positive ω-periodic solutions.

4 Examples

In this section, we illustrate our main results obtained in the previous sections with several examples.

EXAMPLE 4.1. Consider the difference equation

$$x(n+3) = (3 + \sin x(n))x(n) - \lambda\, b(n)x^3(n-9), \tag{6}$$

here $b(n) > 0$ is a 4-periodic sequences,

In fact $2 \leq 3 + \sin(x(n)) \leq 4$ for $n \in [0, 4]$. $f(u) = u^3$ and

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} = 0, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u} = \infty.$$

By Theorems 3.1 and 3.2, (6) has at least one positive ω-periodic solution for sufficiently large $\lambda > 0$ or sufficiently small $\lambda > 0$.

EXAMPLE 4.2. Consider the difference equation

$$x(n+1) = 3x(n) - \sin x(n) - \lambda\, b(n)x(n - \tau(n)), \tag{7}$$

here $b(n) > 0$ and $\tau : Z \to Z$ are ω-periodic sequences.

Clearly, the positive periodic solutions of (7) are the positive periodic solutions of the following difference equation

$$x(n+1) = g(x(n))x(n) - \lambda\, b(n)x(n - \tau(n)), \tag{8}$$

where

$$g(u) = \begin{cases} 3 - \frac{\sin u}{u}, & \text{if } u > 0, \\ 2, & \text{if } u = 0. \end{cases}$$

Note that

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u} = 1, f_\infty = \lim_{u \to \infty} \frac{f(u)}{u} = 1.$$

By Theorem 3.3, (8) has no positive ω-periodic solutions for sufficiently large or small $\lambda > 0$. Hence, (7) has no positive ω-periodic solutions for sufficiently large or small $\lambda > 0$.

Acknowledgement. Supported by the Scientific Research Fund of Hunan Provincial Education Department (09B033) and by Hunan Provincial Natural Science Foundation of China (09JJ3010).

References

[1] L. Rachunek and I. Rachunkova, Strictly increasing solutions of nonautonomous difference equations arising in hydrodynamics, Advances in Difference Equations, Vol. 2010(2010), Article ID 714891, 11 pages.

[2] X. Liu and Y. Liu, On positive periodic solutions of functional difference equations with forcing term and applications, Computers and Mathematics with Applications, 56(9)(2008), 2247–2255.

[3] Y. N. Raffoul, Positive periodic solutions of nonlinear functional difference equations, Electron. J. Differential Equations, 55(2002), 1–8.

[4] D. Jiang, D. O'Regan and R. P. Agarwal, Optimal existence theory for single and multiple positive periodic solutions to functional difference equations, Appl. Math. Comput., 161(2)(2005), 441–462.

[5] Y. Li and L. Zhu, Existence of positive periodic solutions for difference equations with feedback control, Appl. Math. Lett., 18(1)(2005), 61–67.

[6] L. Berezansky and E. Braverman, On existence of positive solutions for linear difference equations with several delays, Advances in Dynamical Systems and Applications, 1(2006), 29–47.

[7] Z. Zeng, Existence of positive periodic solutions for a class of nonautonomous difference equations, Electron J. Differential Equations, Vol. 2006(2006), 1-18.

[8] D. Guo and V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.

[9] P. W. Eloe, Y. Raffoul, D. Reid and K. Yin, Positive solutions of nonlinear functional difference equations, Computers and Mathematics With applications, 42(2001), 639–646.

[10] J. Henderson and S. Lauer, Existence of a positive solution for an n-th order boundary value problem for nonlinear difference equations, Applied and Abstract Analysis, 1(1997), 271–279.

[11] M. Ma and J. Yu, Existence of multiple positive periodic solutions for nonlinear functional difference equations, Journal of Mathematical Analysis and Applications. 305(2)(2005), 483–490.