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Positive Solutions For Singular Boundary Value Problems Involving p-Laplacian Operators*

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Abstract

The existence of positive solutions for singular boundary value problems involving p-Laplacian operators are investigated. By applying the fixed point theorem of cone expansion and compression of norm type, sufficient conditions are established for the existence of positive solutions.

1 Introduction

In this paper, we study the following singular boundary value problem (BVP) involving p-Laplacian operators

$$\begin{cases} (\phi_p(x'))' + a(t)f(x^t, y^t) = 0, \ 0 < t < 1, \\ (\phi_p(y'))' + b(t)g(x^t, y^t) = 0, \ 0 < t < 1, \\ x(t) = \varphi(t), 1 \le t \le 1+\tau, x'(0) = 0, \\ y(t) = \phi(t), 1 \le t \le 1+\tau, y'(0) = 0, \end{cases} \tag{1}$$

where $x^t = x(t+\theta)$, $\theta \in [0, \tau]$, $0 \le \tau < 1$; $\phi_p(\cdot)$ is the p-Laplacian operator; $\varphi, \phi : [1, 1+\tau] \to [0, +\infty)$ are continuous, and $\varphi(1) = \phi(1) = 0$.

For p-Laplacian equations, many results have been obtained, for example see papers [1-4]. But most of them are concerned with ordinary differential equations. Recently, the study of BVP of functional differential equations [5-6] is of significance since they arise and have applications in variational problems of control theory and in other areas of applied mathematics. In this paper, by constructing an integral equation which is equivalent to BVP (1), we study the existence of positive solutions of nonlinear singular BVP of the form (1).

Let $C = C([0, \tau], R)$ be a Banach space with a norm $||\omega||_C = \sup_{0\le\theta\le\tau} |\omega(\theta)|$ and

$$C^+ = \{\omega \in C : \omega(\theta) \ge 0, \theta \in [0, \tau]\}.$$

We assume the following:

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(H_1) $f, g : C^+ \times C^+ \to (0, +\infty)$ are continuous;
(H_2) $a, b : (0, 1) \to [0, +\infty)$ are continuous, and

$$0 < \int_E a(t) \leq \int_0^1 a(t)dt < +\infty,\ 0 < \int_E b(t) \leq \int_0^1 b(t)dt < +\infty,$$

$$0 < \int_0^1 \phi_q(\int_0^s a(r)dr)ds < +\infty,\ 0 < \int_0^1 \phi_q(\int_0^s b(r)dr)ds < +\infty.$$

In this paper, we may choose a $\sigma \in (0, \min\{\frac{1}{4}, \frac{1-\tau}{4}\})$ by (H_2) such that

$$\int_\sigma^{1-\tau-\sigma} a(t)dt > 0 \text{ and } \int_\sigma^{1-\tau-\sigma} b(t)dt > 0.$$

Define $C^* = \{\omega \in C^+ : 0 < \sigma||\omega||_C \leq \omega(\theta), \theta \in [0, \tau]\}$ and $E = \{t \in [0, 1] : 0 \leq t + \theta \leq 1, 0 \leq \theta \leq \tau\} = [0, 1 - \tau]$. Note that for $t \in [\sigma, 1 - \tau - \sigma] \subset E$, we have $x_0^t = y_0^t = 0$.

DEFINITION 1. A function $(x, y) \in C^1[0, 1] \times C^1[0, 1]$ is called a positive solution of BVP (1) if it satisfies the following:
1. (x, y) satisfies BVP(1);
2. $x(t) > 0, y(t) > 0, t \in (0, 1)$; and
3. $(\phi_p(x'), \phi_p(y'))$ is absolutely continuous on $[0, 1]$.

Suppose $(x(t), y(t))$ is a solution of BVP (1). Then

$$\begin{cases} x(t) = \begin{cases} \int_t^1 \phi_q[\int_0^s a(r)f(x^r, y^r)dr]ds, & 0 \leq t \leq 1, \\ \varphi(t), & 1 \leq t \leq 1 + \tau, \end{cases} \\ y(t) = \begin{cases} \int_t^1 \phi_q[\int_0^s b(r)g(x^r, y^r)dr]ds, & 0 \leq t \leq 1, \\ \phi(t), & 1 \leq t \leq 1 + \tau. \end{cases} \end{cases} \tag{2}$$

Suppose that $(x_0(t), y_0(t))$ is the solution of BVP (1) with $f \equiv 0, g \equiv 0$. Then

$$\begin{cases} x_0(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ \varphi(t), & 1 \leq t \leq 1 + \tau, \end{cases} \\ y_0(t) = \begin{cases} 0, & 0 \leq t \leq 1, \\ \phi(t), & 1 \leq t \leq 1 + \tau. \end{cases} \end{cases} \tag{3}$$

If $(x(t), y(t))$ is the solution of BVP (1) and $(u(t), v(t)) = (x(t) - x_0(t), y(t) - y_0(t))$, noting that $(u(t), v(t)) = (x(t), y(t))$ for $0 \leq t \leq 1$, we have from (2) that

$$\begin{cases} u(t) = \begin{cases} \int_t^1 \phi_q[\int_0^s a(r)f(u^r + x_0^r, v^r + y_0^r)dr]ds, & 0 \leq t \leq 1, \\ 0, & 1 \leq t \leq 1 + \tau, \end{cases} \\ v(t) = \begin{cases} \int_t^1 \phi_q[\int_0^s b(r)g(u^r + x_0^r, v^r + y_0^r)dr]ds, & 0 \leq t \leq 1, \\ 0, & 1 \leq t \leq 1 + \tau. \end{cases} \end{cases} \tag{4}$$

Let K be a cone in Banach space $X = C[0, 1 + \tau] \times C[0, 1 + \tau]$ defined by

$$K = \{(u, v) \in X : u(t) \geq 0, v(t) \geq 0, u(t) + v(t) \geq g(t)||(u, v)||, t \in [0, 1]\},$$

where $||(u,v)|| = ||u|| + ||v||, ||u|| = \sup_{t\in[0,b]} |u(t)|, ||v|| = \sup_{t\in[0,b]} |v(t)|$, and

$$g(t) = \begin{cases} 1-t, & 0 \le t \le 1, \\ 0, & 1 \le t \le 1+\tau. \end{cases}$$

Define

$$A(u,v)(t) = \begin{cases} \int_t^1 \phi_q[\int_0^s a(r) f(u^r + x_0^r, v^r + y_0^r) dr] ds, & 0 \le t \le 1, \\ 0, & 1 \le t \le 1+\tau, \end{cases}$$

$$B(u,v)(t) = \begin{cases} \int_t^1 \phi_q[\int_0^s b(r) g(u^r + x_0^r, v^r + y_0^r) dr] ds, & 0 \le t \le 1, \\ 0, & 1 \le t \le 1+\tau, \end{cases}$$

and

$$\Phi(u,v)(t) = (A(u,v)(t), B(u,v)(t)),\ 0 \le t \le 1+\tau. \tag{5}$$

Under assumptions (H_1) and (H_2), BVP (1) has a solution if and only if Φ has a fixed point (u,v), that is, $\Phi(u,v) = (u,v)$.

The following lemma will play an important role in the proof of our results and can be found in the book [7].

LEMMA 1. Assume that X is a Banach space and $K \subset X$ is a cone in X; Ω_1, Ω_2 are open subsets of X, and $0 \in \overline{\Omega}_1 \subset \Omega_2$. Furthermore, let $\Phi : K \bigcap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator satisfying one of the following conditions:

(i) $||\Phi(x)|| \le ||x||,\ \forall\, x \in K \bigcap \partial\Omega_1;\ ||\Phi(x)|| \ge ||x||,\ \forall\, x \in K \bigcap \partial\Omega_2;$

(ii) $||\Phi(x)|| \le ||x||,\ \forall\, x \in K \bigcap \partial\Omega_2;\ ||\Phi(x)|| \ge ||x||,\ \forall\, x \in K \bigcap \partial\Omega_1.$

Then there is a fixed point of Φ in $K \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$.

LEMMA 2. The map $\Phi : X \to X$ in (5) is completely continuous and $\Phi(K) \subset K$.

The proof of Lemma 2 can be found in [5-6].

2 Main Results

In the sequel, we let

$$f_0 := \lim_{(||\omega_1||_C + ||\omega_2||_C) \to 0} \frac{f(\omega_1, \omega_2)}{(||\omega_1||_C + ||\omega_2||_C)^{p-1}};$$

$$f_0^* := \lim_{\omega_1,\omega_2 \in C^*, (||\omega_1||_C + ||\omega_2||_C) \to 0} \frac{f(\omega_1, \omega_2)}{(||\omega_1||_C + ||\omega_2||_C)^{p-1}};$$

$$f_\infty := \lim_{(||\omega_1||_C + ||\omega_2||_C) \to \infty} \frac{f(\omega_1, \omega_2)}{(||\omega_1||_C + ||\omega_2||_C)^{p-1}};$$

$$f_\infty^* := \lim_{\omega_1,\omega_2 \in C^*, (||\omega_1||_C + ||\omega_2||_C) \to \infty} \frac{f(\omega_1, \omega_2)}{(||\omega_1||_C + ||\omega_2||_C)^{p-1}};$$

$$g_0 := \lim_{(||\omega_1||_C+||\omega_2||_C)\to 0} \frac{f(\omega_1,\omega_2)}{(||\omega_1||_C+||\omega_2||_C)^{p-1}};$$

$$g_0^* := \lim_{\omega_1,\omega_2\in C^*,(||\omega_1||_C+||\omega_2||_C)\to 0} \frac{f(\omega_1,\omega_2)}{(||\omega_1||_C+||\omega_2||_C)^{p-1}};$$

$$g_\infty := \lim_{(||\omega_1||_C+||\omega_2||_C)\to \infty} \frac{g(\omega_1,\omega_2)}{(||\omega_1||_C+||\omega_2||_C)^{p-1}};$$

and

$$g_\infty^* := \lim_{\omega_1,\omega_2\in C^*,(||\omega_1||_C+||\omega_2||_C)\to \infty} \frac{g(\omega_1,\omega_2)}{(||\omega_1||_C+||\omega_2||_C)^{p-1}}.$$

THEOREM 1. Assume (H_1) and (H_2) hold. Then BVP (1) has at least one positive solution if one of the following conditions is satisfied:

(H_3) $f_0 = 0,\ f_\infty^* = +\infty,\ g_0 = 0,\ \varphi(t) = \phi(t) \equiv 0,\ t \in [1, 1+\tau]$; or

(H_4) $f_0^* = +\infty,\ f_\infty = 0,\ g_\infty = 0.$

PROOF. Suppose that (H_3) is satisfied. By $\phi(t) \equiv 0, \varphi(t) \equiv 0, t \in [1, 1+\tau]$, we know $x_0^t = y_0^t = 0, t \in [0, 1+\tau]$. Since $f_0 = 0$, for $\varepsilon > 0$ (we choose ε satisfying $\varepsilon \int_0^1 \phi_q[\int_0^s a(r)dr]ds \leq \frac{1}{2}$), there is a $\rho_1 > 0$ such that

$$f(\omega_1,\omega_2) \leq (\varepsilon(||\omega_1||_C+||\omega_2||_C))^{p-1}, 0 \leq ||\omega_1||_C+||\omega_2||_C \leq \rho_1.$$

Define

$$\Omega_1 = \{(u,v) \in X : ||(u,v)|| < \rho_1\}.$$

For $(u,v) \in \partial\Omega_1 \cap K$, we deduce that $||u^r||_C + ||v^r||_C \leq \rho_1$ for $r \in [0,1]$ and thus

$$\begin{aligned} ||A(u,v)|| &= \int_0^1 \phi_q[\int_0^s a(r)f(u^r,v^r)dr]ds \\ &\leq \int_0^1 \phi_q[\int_0^s a(r)(\varepsilon(||u^r||+||v^r||))^{p-1}dr]ds \\ &\leq \varepsilon(||u||+||v||)\int_0^1 \phi_q[\int_0^s a(r)dr]ds \\ &\leq \frac{1}{2}(||u||+||v||). \end{aligned}$$

Similarly, we have $B(u,v) \leq \frac{1}{2}(||u||+||v||)$. This implies

$$||\Phi(u,v)|| = ||A(u,v)|| + ||B(u,v)|| \leq ||(u,v)||, (u,v) \in \partial\Omega_1 \cap K.$$

On the other hand, since $f_\infty^* = +\infty$, for $M > 0$ we can choose M satisfying $M\sigma \int_\sigma^{1-\tau-\sigma} \phi_q[\int_\sigma^s a(r)dr]ds \geq 1$, there exists a $\rho_2 > \rho_1$ such that

$$f(\omega_1,\omega_2) \geq (M(||\omega_1||_C+||\omega_2||_C))^{p-1},\ \omega_1,\omega_2 \in C^*,\ ||\omega_1||_C+||\omega_2||_C \geq \rho_2.$$

Define

$$\Omega_2 = \{(u,v) \in X : ||(u,v)|| < \rho_2\}.$$

For $(u,v)\in\partial\Omega_2\cap K$, we deduce

$$\sigma||u^r||_C\leq\sigma||u||\leq g(t)||u||\leq u(t),\ t\in[\sigma,1-\sigma],\ r\in[0,1],$$

$$\sigma||v^r||_C\leq\sigma||v||\leq g(t)||v||\leq v(t),\ t\in[\sigma,1-\sigma],\ r\in[0,1],$$

which implies that $u^r,v^r\in C^*$ for $r\in[\sigma,1-\tau-\sigma]$ and

$$||u^r||_C\geq\sigma||u||=\sigma\rho_2,\ ||v^r||_C\geq\sigma||v||=\sigma\rho_2,\ r\in[\sigma,1-\tau-\sigma].$$

Thus, for $(u,v)\in\partial\Omega_2\cap K$, we have

$$\begin{aligned}||A(u,v)|| &= \int_0^1\phi_q[\int_0^s a(r)f(u^r,v^r)dr]ds\\ &\geq \int_\sigma^{1-\tau-\sigma}\phi_q[\int_\sigma^s a(r)(M(||u^r||_C+||v^r||_C))^{p-1}dr]ds\\ &\geq M\sigma(||u||+||v||)\int_\sigma^{1-\tau-\sigma}\phi_q[\int_\sigma^s a(r)dr]ds\\ &\geq ||u||+||v||\\ &= ||(u,v)||.\end{aligned}$$

That is,

$$||\Phi(u,v)||\geq||(u,v)||,(u,v)\in\partial\Omega_2\cap K.$$

According to the first part of Lemma 1, it follows that Φ has a fixed point$(u,v)\in K\bigcap(\overline{\Omega}_2\setminus\Omega_1)$.

Now, suppose that (H_4) is satisfied. Since $f_0^*=+\infty$, for $M>0$ (choose M satisfying $M\sigma\int_\sigma^{1-\tau-\sigma}\phi_q[\int_\sigma^s a(r)dr]ds\geq1$), there exists a $\rho_1>0$ such that

$$f(\omega_1,\omega_2)\geq(M(||\omega_1||_C+||\omega_2||_C))^{p-1},\omega_1,\omega_2\in C^*||\omega_1||_C+||\omega_2||_C\leq\rho_1.$$

Define

$$\Omega_1=\{(u,v)\in X:||(u,v)||<\rho_1\}.$$

For $(u,v)\in\partial\Omega_1\cap K$, we deduce

$$\sigma||u^r||_C\leq\sigma||u||\leq g(t)||u||\leq u(t),t\in[\sigma,1-\sigma],r\in[0,1],$$

$$\sigma||v^r||_C\leq\sigma||v||\leq g(t)||v||\leq v(t),t\in[\sigma,1-\sigma],r\in[0,1],$$

which implies that $u^r,v^r\in C^*$ for $r\in[\sigma,1-\tau-\sigma]$ and

$$||u^r||_C\geq\sigma||u||=\sigma\rho_1,\ ||v^r||_C\geq\sigma||v||=\sigma\rho_1,\ r\in[\sigma,1-\tau-\sigma]. \tag{6}$$

For $r \in [\sigma.1-\tau-\sigma]$, we have $x_0^r = y_0^r = 0$. Thus, for $(u,v) \in \partial\Omega_2 \cap K$, we have

$$\begin{aligned}
||A(u,v)|| &= \int_0^1 \phi_q[\int_0^s a(r) f(u^r, v^r) dr] ds \\
&\geq \int_\sigma^{1-\tau-\sigma} \phi_q[\int_\sigma^s a(r)(M(||u^r||_C + ||v^r||_C))^{p-1} dr] ds \\
&\geq M\sigma(||u|| + ||v||) \int_\sigma^{1-\tau-\sigma} \phi_q[\int_\sigma^s a(r) dr] ds \\
&\geq ||u|| + ||v|| \\
&= ||(u,v)||,
\end{aligned}$$

which implies $||\Phi(u,v)|| \geq ||(u,v)||, \forall (u,v) \in \partial\Omega_1 \cap K$.

On the other hand, since $f_\infty = 0$, for $\forall\ \varepsilon > 0,\ \exists N > \rho_1$ such that

$$f(\omega_1,\omega_2) \leq (\varepsilon(||\omega_1||_C + ||\omega_2||_C))^{p-1},\ ||\omega_1||_C + ||\omega_2||_C > N.$$

Choose a positive constant ρ_2 such that

$$\rho_2 > 1 + \max\{f^{q-1}(\omega_1,\omega_2) : 0 \leq ||\omega_1||_C + ||\omega_2||_C \leq N + ||u_0|| + ||v_0||\} \phi_q[\int_0^1 (a(r)+b(r)) dr].$$

Define

$$\Omega_2 = \{(u,v) \in X : ||(u,v)|| < \rho_2\}.$$

For $(u,v) \in \partial\Omega_2 \cap K$, we have from the facts: $x_0(t) \geq 0, u(t) \geq 0, y_0(t) \geq 0, v(t) \geq 0, t \in [0, 1+\tau]$, that for $r \in [0,1]$,

$$||u^r + x_0^r||_C + ||v^r + y_0^r||_C \geq ||u^r||_C + ||v^r||_C > N, \quad \text{for } ||u^r||_C + ||v^r||_C > N,$$

and

$$\begin{aligned}
||u^r + x_0^r||_C + ||v^r + y_0^r||_C &\leq ||u^r||_C + ||x_0^r||_C + ||v^r||_C + ||y_0^r||_C \\
&\leq N + ||x_0|| + ||y_0||,
\end{aligned}$$

for $||u^r||_C + ||v^r||_C \leq N$. Let

$$\alpha := \max\{f^{q-1}(\omega_1,\omega_2) : 0 \leq ||\omega_1||_C + ||\omega_2||_C \leq N + ||x_0|| + ||y_0||\}.$$

Thus, for ε satisfying $0 < \varepsilon(1 + ||x_0|| + ||y_0||)\phi_q[\int_0^1 a(r)dr] < \frac{1}{2}$, we have

$$\begin{aligned}
||A(u,v)|| &= \int_0^1 \phi_q[\int_0^s a(r)f(u^r + x_0^r, v^r + y_0^r)dr]ds \\
&\leq \int_0^1 \phi_q[\int_0^1 a(r)f(u^r + x_0^r, v^r + y_0^r)dr]ds \\
&= \phi_q[\int_{||u^r||_C+||v^r||_C>N} a(r)f(u^r + x_0^r, v^r + y_0^r)dr \\
&\quad + \int_{0\leq||u^r||_C+||v^r||_C\leq N} a(r)f(u^r + x_0^r, v^r + y_0^r)dr] \\
&\leq \max\{\varepsilon(||u^r + x_0^r||_C + ||v^r + y_0^r||_C), \alpha\}\phi_q[\int_0^1 a(r)dr] \\
&\leq \max\{\varepsilon(||u + x_0|| + ||v + y_0||), \alpha\}\phi_q[\int_0^1 a(r)dr] \\
&\leq \max\{\frac{1}{2}(||u|| + ||v||) + \frac{1}{2}, \alpha\phi_q[\int_0^1 a(r)dr]\} \\
&< \frac{1}{2}(||u|| + ||v||) \\
&= \frac{1}{2}\rho_2.
\end{aligned}$$

Similarly, we have $B(u,v) \leq \frac{1}{2}(||u|| + ||v||)$. This implies

$$||\Phi(u,v)|| = ||A(u,v)|| + ||B(u,v)|| \leq ||(u,v)||, (u,v) \in \partial\Omega_2 \cap K.$$

According to the second part of Lemma 1, it follows that Φ has a fixed point$(u,v) \in K\bigcap(\overline{\Omega}_2 \setminus \Omega_1)$.

Suppose that $(u(t),v(t))$ is the fixed point of Φ in $K\bigcap(\overline{\Omega}_2 \setminus \Omega_1)$, then $(x(t),y(t)) = (u(t)+x_0(t), v(t)+y_0(t))$ is a positive solution of BVP (1). This completes the proof.

Similarly, we have the next theorem.

THEOREM 2. Assume (H_1) and (H_2) hold. Then BVP (1) has at least a positive solution if one of the following conditions is satisfied:

(H_3') $f_0 = 0;\ g_\infty^* = +\infty;\ g_0 = 0;\ \varphi(t) = \phi(t) \equiv 0,\ t \in [1, 1+\tau]$, or

(H_4') $f_\infty = 0;\ g_\infty = 0;\ g_0^* = +\infty$.

In what follows, we shall consider the existence of multiple positive solutions for BVP (1).

THEOREM 3. Assume $(H_1),(H_2)$ hold and the following conditions are satisfied:

(H_5) $f_0^* = +\infty;\ f_\infty^* = +\infty$;

(H_6) $\exists$ a $p_1 > 0$ such that for $\forall\ 0 \leq ||\omega_1||_C + ||\omega_2||_C \leq p_1 + p_0$, one has

$$f(\omega_1,\omega_2) \leq (l_1p_1)^{p-1} \text{ and } g(\omega_1,\omega_2) \leq (l_1p_1)^{p-1},$$

where $p_0 = \max\limits_{1\leq t\leq 1+\tau}\{\phi(t),\varphi(t)\},\ l_1 = \{2\int_0^1 \phi_q[\int_0^s a(r)dr]ds\}^{-1}$.

Then BVP (1) has at least two positive solutions.

PROOF. By (H_5), there exists a $\rho_1 : 0 < \rho_1 < p_1$ such that

$$f(\omega_1, \omega_2) \geq (M(||\omega_1||_C + ||\omega_2||_C))^{p-1}, \ ||\omega_1||_C + ||\omega_2||_C \leq \rho_1, \ \omega_1, \omega_2 \in C^*,$$

where M satisfies $M\sigma \int_{1-\sigma}^{1} \phi_q[\int_{\sigma}^{1-\tau-\sigma} a(r)dr]ds \geq 1$. Define

$$\Omega_1 = \{(u, v) \in X : ||(u, v)|| < \rho_1\}.$$

For $(u, v) \in \partial\Omega_1 \cap K$, similar to (6) one has $u^r, v^r \in C^*$ and

$$\rho_1 \geq ||u^r||_C + ||v^r||_C \geq \sigma(||u|| + ||v||) = \sigma\rho_1, \ r \in [\sigma, 1 - \tau - \sigma].$$

Hence, we obtain an analogous inequality:

$$||\Phi(u, v)|| = ||A(u, v)|| + ||B(u, v)|| \geq ||(u, v)||, (u, v) \in \partial\Omega_1 \cap K.$$

Similarly, there exists a $\rho_3 > p_1$ such that

$$f(\omega_1, \omega_2) \geq (M(||\omega_1||_C + ||\omega_2||_C))^{p-1}, \ ||\omega_1||_C + ||\omega_2||_C \geq \sigma\rho_3, \ \omega_1, \omega_2 \in C^*,$$

M is chosen as above. Define

$$\Omega_3 = \{(u, v) \in X : ||(u, v)|| < \rho_3\}.$$

For $(u, v) \in \partial\Omega_3 \cap K$, one has $u^r, v^r \in C^*$ and

$$||u^r||_C + ||v^r||_C \geq \sigma(||u|| + ||v||) = \sigma\rho_3, \ r \in [\sigma, 1 - \tau - \sigma].$$

Furthermore, we have

$$||\Phi(u, v)|| = ||A(u, v)|| + ||B(u, v)|| \geq ||(u, v)||, (u, v) \in \partial\Omega_3 \cap K.$$

By (H_6), let $\rho_2 = p_1$, define $\Omega_2 = \{(u, v) \in X : ||(u, v)|| < \rho_2\}$. For $(u, v) \in \partial\Omega_2 \cap K$, one has

$$\begin{aligned} ||A(u, v)|| &= \int_0^1 \phi_q[\int_0^s a(r) f(u^r + x_0^r, v^r + y_0^r)dr]ds \\ &\leq l_1 p_1 \int_0^1 \phi_q[\int_0^s a(r)dr]ds \\ &= \frac{1}{2} p_1 \\ &= \frac{1}{2} \rho_2 \\ &= \frac{1}{2} ||(u, v)||. \end{aligned}$$

Similarly, we have $B(u, v) \leq \frac{1}{2}(||u|| + ||v||)$. This implies

$$||\Phi(u, v)|| = ||A(u, v)|| + ||B(u, v)|| \leq ||(u, v)||, (u, v) \in \partial\Omega_2 \cap K.$$

According to Lemma 1, it follows that Φ has two fixed points, that is to say, BVP (1) has at least two positive solutions. This completes the proof.

From above, the following theorems are obvious.

THEOREM 4. Assume (H_1) and (H_2) hold and the following conditions are satisfied:

(H_7) $f_0 = 0,\ f_\infty = 0,\ g_0 = 0,\ g_\infty = 0,\ \phi(t) = \varphi(t) \equiv 0.$

(H_8) $\exists$ a $p_2 > 0$ such that for $\forall\ \sigma p_2 \leq ||\omega_1||_C + ||\omega_2||_C \leq p_2$, one has

$$f(\omega_1, \omega_2) \geq (l_2 p_2)^{p-1},$$

where $l_2 = \left\{\int_0^{1-\tau-\sigma} \phi_q[\int_0^s a(r)dr]dt\right\}^{-1}$.

Then BVP (1) has at least two positive solutions.

THEOREM 5. Assume (H_1) and (H_2) hold and the following conditions are satisfied:

(H_5') $g_0^* = +\infty;\ g_\infty^* = +\infty;$

(H_6) $\exists$ a $p_1 > 0$ such that for $\forall\ 0 \leq ||\omega_1||_C + ||\omega_2||_C \leq p_1 + p_0$, one has

$$f(\omega_1, \omega_2) \leq (l_1 p_1)^{p-1} \text{ and } g(\omega_1, \omega_2) \leq (l_1 p_1)^{p-1},$$

where $p_0 = \max\limits_{1 \leq t \leq 1+\tau}\{\phi(t), \varphi(t)\}$ and $l_1 = \left\{2\int_0^1 \phi_q[\int_0^s a(r)dr]ds\right\}^{-1}$.

Then BVP (1) has at least two positive solutions.

THEOREM 6. Assume (H_1) and (H_2) hold and the following conditions are satisfied:

(H_7) $f_0 = 0;\ f_\infty = 0;\ g_0 = 0;\ g_\infty = 0;\ \phi(t) = \varphi(t) \equiv 0.$

(H_9') $\exists$ a $p_2 > 0$ such that for $\forall\ \sigma p_2 \leq ||\omega_1||_C + ||\omega_2||_C \leq p_2$, one has

$$g(\omega_1, \omega_2) \geq (l_2 p_2)^{p-1},$$

where $l_2 = \left\{\int_0^{1-\tau-\sigma} \phi_q[\int_0^s b(r)dr]dt\right\}^{-1}$.

Then BVP (1) has at least two positive solutions.

3 Examples

We have several examples.

EXAMPLE 1. Consider BVP

$$\begin{cases} (\phi_p(x'))'^{\frac{1}{4}}(t+\frac{1}{3}) + y^{\frac{1}{3}}(t+\frac{1}{3}) = 0,\ 0 < t < 1, \\ (\phi_p(y'))'^{\frac{1}{3}}(t+\frac{1}{3}) + y^{\frac{1}{4}}(t+\frac{1}{3}) = 0,\ 0 < t < 1, \\ x(t) = 0, 1 \leq t \leq 1+\frac{1}{3}, x'(0) = 0, \\ y(t) = 0, 1 \leq t \leq 1+\frac{1}{3}, y'(0) = 0. \end{cases} \tag{7}$$

Here, $a(t) = b(t) = 1;\ x^t = x(t+\theta) \equiv x(t+\frac{1}{3}),\ y^t = y(t+\theta) \equiv y(t+\frac{1}{3});\ \tau = \frac{1}{3};\ p = \frac{9}{8}$, and

$$f(\omega_1, \omega_2) = \omega_1^{\frac{1}{4}}(\frac{1}{3}) + \omega_2^{\frac{1}{3}}(\frac{1}{3});\ g(\omega_1, \omega_2) = \omega_1^{\frac{1}{3}}(\frac{1}{3}) + \omega_2^{\frac{1}{4}}(\frac{1}{3}).$$

As $||\omega_1||_C + ||\omega_2||_C \to 0$, we have

$$\begin{aligned}
\frac{f(\omega_1,\omega_1)}{(||\omega_1||_C+||\omega_2||_C)^{p-1}} &= \frac{\omega_1^{\frac{1}{4}}(\frac{1}{3})+\omega_2^{\frac{1}{3}}(\frac{1}{3})}{(||\omega_1||_C+||\omega_2||_C)^{\frac{1}{8}}} \\
&\leq \frac{||\omega_1||_C^{\frac{1}{4}}+||\omega_2||_C^{\frac{1}{3}}}{(||\omega_1||_C+||\omega_2||_C)^{\frac{1}{8}}} \\
&\leq (||\omega_1||_C+||\omega_2||_C)^{\frac{1}{8}} \\
&\to 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{g(\omega_1,\omega_1)}{(||\omega_1||_C+||\omega_2||_C)^{p-1}} &= \frac{\omega_1^{\frac{1}{3}}(\frac{1}{3})+\omega_2^{\frac{1}{4}}(\frac{1}{3})}{(||\omega_1||_C+||\omega_2||_C)^{\frac{1}{8}}} \\
&\leq \frac{||\omega_1||_C^{\frac{1}{3}}+||\omega_2||_C^{\frac{1}{4}}}{(||\omega_1||_C+||\omega_2||_C)^{\frac{1}{8}}} \\
&\leq (||\omega_1||_C+||\omega_2||_C)^{\frac{1}{8}} \\
&\to 0,
\end{aligned}$$

that is to say $f_0 = 0$ and $g_0 = 0$ hold.

On the other hand, suppose $\omega_1, \omega_2 \in C^*$. Then $\omega_1(\theta) \geq \sigma||\omega_1||_C, \omega_2(\theta) \geq \sigma||\omega_2||_C$. As $\omega_1, \omega_2 \in C^*, ||\omega_1||_C + ||\omega_2||_C \to \infty$, we get

$$\begin{aligned}
\frac{f(\omega_1,\omega_1)}{(||\omega_1||_C+||\omega_2||_C)^{p-1}} &= \frac{\omega_1^{\frac{1}{4}}(\frac{1}{3})+\omega_2^{\frac{1}{3}}(\frac{1}{3})}{(||\omega_1||_C+||\omega_2||_C)^{\frac{1}{8}}} \\
&\geq \frac{(\sigma||\omega_1||_C)^{\frac{1}{4}}+(\sigma||\omega_2||_C)^{\frac{1}{3}}}{(||\omega_1||_C+||\omega_2||_C)^{\frac{1}{8}}} \\
&\geq \sigma^{\frac{1}{3}}(||\omega_1||_C+||\omega_2||_C)^{\frac{1}{8}} \\
&\to +\infty,
\end{aligned}$$

which means that $f_0^* = +\infty$ holds. According to Theorem 1, it follows that BVP (7) has at least one positive solution.

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