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Coefficient Estimates Of Functions In The Class Concerning With Spirallike Functions*

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Abstract

For analytic functions $f(z)$ normalized by $f(0) = 0$ and $f'(0) = 1$ in the open unit disk $\mathbb{U}$, a new subclass $\mathcal{S}_\alpha$ of $f(z)$ concerning with spirallike functions in $\mathbb{U}$ is introduced. The object of the present paper is to discuss an extremal function for the class $\mathcal{S}_\alpha$ and coefficient estimates of functions $f(z)$ belonging to the class $\mathcal{S}_\alpha$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}\,;\, |z| < 1\}$. Let $\mathcal{S}^*(\alpha)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U})$$

for some real α $(0 \leqq \alpha < 1)$. A function $f(z)$ in the class $\mathcal{S}^*(\alpha)$ is said to be starlike of order α in $\mathbb{U}$. Further, if a function $f(z) \in \mathcal{A}$ satisfies

$$\mathrm{Re}\left(e^{i\lambda}\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbb{U})$$

for some real λ $(|\lambda| < \frac{\pi}{2})$, then we say that $f(z)$ is spirallike in $\mathbb{U}$. We also note that a spirallike function in $\mathbb{U}$ is univalent in $\mathbb{U}$ (cf. Duren [2]).

If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\mathrm{Re}\left(\frac{1}{\alpha}\frac{zf'(z)}{f(z)}\right) > 1 \qquad (z \in \mathbb{U}) \tag{2}$$

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for some complex number α ($|\alpha - \frac{1}{2}| < \frac{1}{2}$), then we say that $f(z) \in \mathcal{S}_\alpha$. This class $\mathcal{S}_\alpha$ was recently introduced by Hamai, Hayami and Owa [3]. If $\alpha = |\alpha|e^{i\varphi}$, then the condition (2) is equivalent to

$$\mathrm{Re}\left(e^{-i\varphi}\frac{zf'(z)}{f(z)}\right) > |\alpha| \qquad (z \in \mathbb{U}).$$

Therefore, we note that a function $f(z) \in \mathcal{S}_\alpha$ is spirallike in $\mathbb{U}$ which implies that $f(z)$ is univalent in $\mathbb{U}$. Further, if $0 < \alpha < 1$, then $f(z) \in \mathcal{S}_\alpha$ is starlike of order α (cf. Robertson [4]).

Let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \tag{3}$$

which are analytic in $\mathbb{U}$ and satisfy

$$\mathrm{Re}\ p(z) > 0 \qquad (z \in \mathbb{U}).$$

Then we say that $p(z) \in \mathcal{P}$ is the Carathéodory function (cf. Caratéodory [1] or Duren [2]).

REMARK 1. Let us consider a function $f(z) \in \mathcal{A}$ which satisfies

$$\left|\frac{f(z)}{zf'(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2|\alpha|} \qquad (z \in \mathbb{U}) \tag{4}$$

for $|\alpha - \frac{1}{2}| < \frac{1}{2}$. If we write that $F(z) = \frac{zf'(z)}{f(z)}$, then the inequality (4) can be written by

$$\left|\frac{2\alpha - F(z)}{F(z)}\right| < 1 \qquad (z \in \mathbb{U}).$$

This implies that

$$\alpha\overline{F(z)} + \bar{\alpha}F(z) > 2|\alpha|^2 \qquad (z \in \mathbb{U}).$$

It follows that

$$\left(\frac{F(z)}{\alpha}\right) + \overline{\left(\frac{F(z)}{\alpha}\right)} > 2 \qquad (z \in \mathbb{U}).$$

Therefore, the inequality (4) is equivalent to

$$\mathrm{Re}\left(\frac{1}{\alpha}\frac{zf'(z)}{f(z)}\right) > 1 \qquad (z \in \mathbb{U}).$$

2 Coefficient Estimates

In this section, we discuss the coefficient estimates of a_n for $f(z) \in \mathcal{S}_\alpha$. To establish our results, we need the following lemma due to Carathéodory [1].

LEMMA 1. If a function $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in P$, then

$$|c_k| \leq 2 \qquad (k = 1, 2, 3, ...)$$

with equality for

$$p(z) = \frac{1+z}{1-z}.$$

Now, we introduce the following theorem.

THEOREM 1. The extremal function $f(z)$ for the class $\mathcal{S}_\alpha$ is defined by

$$f(z) = \frac{z}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}}. \tag{5}$$

PROOF. From the definition of the class $\mathcal{S}_\alpha$, we have that

$$\mathrm{Re}\left(\frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1\right) > 0.$$

Moreover, it is clear that

$$\mathrm{Re}\left(\frac{1}{\alpha}\right) > 1 \qquad \left(\left|\alpha - \frac{1}{2}\right| < \frac{1}{2}\right).$$

Then, if the function $F(z)$ is defined by

$$F(z) = \frac{\frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1 - i\mathrm{Im}\left(\frac{1}{\alpha}\right)}{\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1},$$

we see that $\mathrm{Re}\, F(z) > 0$ and $F(0) = 1$, so that, $F(z) \in \mathcal{P}$. Therefore, if $F(z)$ satisfies

$$F(z) = \frac{\frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1 - i\mathrm{Im}\left(\frac{1}{\alpha}\right)}{\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1} = \frac{1+z}{1-z},$$

then $F(z)$ satisfies the equality in Lemma 1. Thus, the function $f(z)$ given by the above is said to be the extremal function for the class $\mathcal{S}_\alpha$. Note that

$$\frac{f'(z)}{f(z)} - \frac{1}{z} = 2\alpha\left(\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1\right)\frac{1}{1-z}.$$

Integrating both sides from 0 to z on t, we have that

$$\int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t}\right)dt = 2\alpha\left(\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1\right)\int_0^z \frac{1}{1-t}dt,$$

which implies that

$$\log\frac{f(z)}{z} = \log\frac{1}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}}.$$

Therefore, we obtain that

$$f(z) = \frac{z}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}}.$$

This is the extremal function of the class $\mathcal{S}_\alpha$.

Next, we discuss the coefficient estimates of $f(z)$ belonging to the class $\mathcal{S}_\alpha$.

THEOREM 2. If a function $f(z) \in \mathcal{S}_\alpha$, then

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)) \qquad (n = 2, 3, 4, ...).$$

Equality holds true for $f(z)$ given by (5) with real $\alpha \in (0, 1)$.

PROOF. By using the same method given in the proof of Theorem 1, if we set $F(z)$ that

$$F(z) = \frac{\frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1 - i\mathrm{Im}\big(\frac{1}{\alpha}\big)}{\mathrm{Re}\big(\frac{1}{\alpha}\big) - 1}, \tag{6}$$

then it is clear that $F(z) \in \mathcal{P}$. Letting

$$F(z) = 1 + c_1 z + c_2 z^2 + \cdots,$$

Lemma 1 gives us that

$$|c_m| \leq 2 \qquad (m = 1, 2, 3, ...).$$

Now, from (6),

$$\left(\mathrm{Re}\left(\frac{1}{\alpha}\right) - 1\right) F(z) = \frac{1}{\alpha}\frac{zf'(z)}{f(z)} - 1 - i\mathrm{Im}\left(\frac{1}{\alpha}\right).$$

Let $\mathrm{Re}(\frac{1}{\alpha}) - 1 = s$ and $1 + i\mathrm{Im}(\frac{1}{\alpha}) = A$. This implies that

$$(\alpha s F(z) + \alpha A) f(z) = zf'(z).$$

Then, the coefficients of z^n in both sides lead to

$$na_n = (\alpha s + \alpha A)a_n + \alpha s(a_{n-1}c_1 + a_{n-2}c_2 + \cdots + a_2 c_{n-2} + c_{n-1}).$$

Therefore, we see that

$$a_n = \frac{\alpha s}{n - \alpha s - \alpha A}(a_{n-1}c_1 + a_{n-2}c_2 + \cdots + a_2 c_{n-2} + c_{n-1}).$$

This shows that

$$|a_n| \quad = \quad \frac{|\alpha\big(\mathrm{Re}(\frac{1}{\alpha}) - 1\big)|}{|n - \alpha\big(\mathrm{Re}(\frac{1}{\alpha}) - 1\big) - \alpha\big(1 + i\mathrm{Im}(\frac{1}{\alpha})\big)|}|a_{n-1}c_1 + a_{n-2}c_2 + \cdots + a_2 c_{n-2} + c_{n-1}|$$

$$\begin{aligned}
&= \frac{\cos(\arg(\alpha)) - |\alpha|}{n-1}|a_{n-1}c_1 + a_{n-2}c_2 + \cdots + a_2c_{n-2} + c_{n-1}| \\
&\leq \frac{\cos(\arg(\alpha)) - |\alpha|}{n-1}(|a_{n-1}||c_1| + |a_{n-2}||c_2| + \cdots + |a_2||c_{n-2}| + |c_{n-1}|) \\
&\leq \frac{\cos(\arg(\alpha)) - |\alpha|}{n-1}(2|a_{n-1}| + 2|a_{n-2}| + \cdots + 2|a_2| + 2) \\
&= \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{n-1}\sum_{k=1}^{n-1}|a_k| \qquad (|a_1| = 1).
\end{aligned}$$

To prove that

$$|a_n| \leq \frac{1}{(n-1)!}\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)),$$

we need to show that

$$|a_n| \leq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{n-1}\sum_{k=1}^{n-1}|a_k| \leq \frac{\prod_{k=1}^{n-1}(2(\cos(\arg(\alpha)) - |\alpha|) + (k-1))}{(n-1)!}. \tag{7}$$

Now, we use the mathematical induction for the proof. When $n = 2$, we see that

$$|a_2| \leq 2(\cos(\arg(\alpha) - |\alpha|)\,|a_1| = 2(\cos(\arg(\alpha) - |\alpha|).$$

Therefore, the assertion is holds true for $n = 2$. Next, we assume that the proposition is true for $n = 2, 3, 4, ..., m-1$. For $n = m$, we obtain that

$$\begin{aligned}
|a_m| &\leq \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1}\sum_{k=1}^{m-1}|a_k| \\
&= \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1}\Big(\sum_{k=1}^{m-2}|a_k| + |a_{m-1}|\Big) \\
&= \frac{m-2}{m-1}\frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-2}\sum_{k=1}^{m-2}|a_k| + \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1}|a_{m-1}| \\
&\leq \frac{m-2}{(m-1)!}\prod_{k=1}^{m-2}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1) \\
&\quad + \frac{2(\cos(\arg(\alpha)) - |\alpha|)}{m-1}\frac{1}{(m-2)!}\prod_{k=1}^{m-2}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1) \\
&= \frac{1}{(m-1)!}\left\{\prod_{k=1}^{m-2}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1)\right\}(m - 2 + 2\left(\cos\left(\arg(\alpha)\right) - |\alpha|\right))\big) \\
&= \frac{1}{(m-1)!}\prod_{k=1}^{m-1}(2(\cos(\arg(\alpha)) - |\alpha|) + k - 1).
\end{aligned}$$

Thus the inequality (7) is true for $n = m$. By the mathematical induction, we prove that

$$|a_n| \leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + (k-1)) \qquad (n = 2, 3, 4, ...).$$

For the equality, we consider the extremal function $f(z)$ given by Theorem 1. Since

$$f(z) = \frac{z}{(1-z)^{2\alpha(\mathrm{Re}(\frac{1}{\alpha})-1)}},$$

if we let

$$2\alpha \left(\mathrm{Re} \left(\frac{1}{\alpha} \right) - 1 \right) = j,$$

then $f(z)$ becomes that

$$f(z) = z(1-z)^{-j} = z \left(\sum_{n=0}^{\infty} \binom{-j}{n} (-z)^n \right) = z + \sum_{n=2}^{\infty} \frac{j(j+1)\cdots(j+n-2)}{(n-1)!} z^n.$$

From the above, we obtain

$$a_n = \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2\alpha \left(\mathrm{Re} \left(\frac{1}{\alpha} \right) - 1 \right) + k - 1).$$

For $n = 2$,

$$|a_2| = 2|\alpha| \left| \mathrm{Re}(\frac{1}{\alpha}) - 1 \right| = 2(\cos(\arg(\alpha)) - |\alpha|).$$

Furthermore, for $n \geq 3$, we have that

$$\begin{aligned} |a_n| &= \left| \frac{1}{(n-1)!} \prod_{k=1}^{n-1} \big(2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) + k - 1\big) \right| \\ &= \frac{1}{(n-1)!} \prod_{k=1}^{n-1} \left| 2\alpha(\mathrm{Re}(\frac{1}{\alpha}) - 1) + k - 1 \right| \\ &\leq \frac{1}{(n-1)!} \prod_{k=1}^{n-1} (2(\cos(\arg(\alpha)) - |\alpha|) + k - 1). \end{aligned}$$

Equality holds true for some real α $(0 < \alpha < 1)$. This completes the proof of Theorem 2.

EXAMPLE 1. Let $\alpha = \frac{1}{2} + \frac{1}{4}i$ in (5). Then we have that

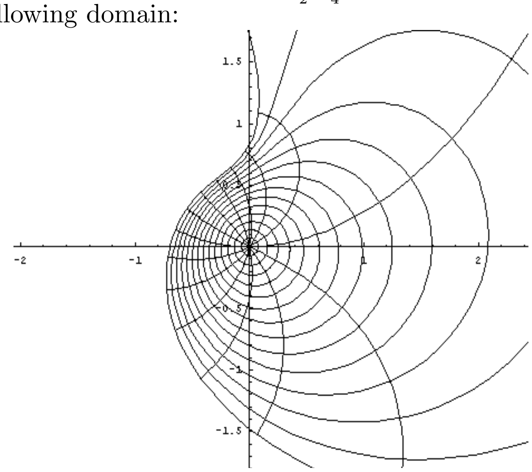
$$f(z) = \frac{z}{(1-z)^{\frac{6+3i}{10}}}.$$

This function $f(z)$ satisfies

$$\mathrm{Re} \left(\frac{1}{\alpha} \frac{zf'(z)}{f(z)} \right) = \mathrm{Re} \left\{ \frac{8-4i}{5} \left(1 + \frac{(6+3i)z}{10(1-z)} \right) \right\}$$

$$= \frac{8}{5} + \frac{6}{5}\mathrm{Re}\left(\frac{z}{1-z}\right) > \frac{8}{5} - \frac{3}{5} = 1.$$

Thus we see that $f(z) \in \mathcal{S}_{\frac{1}{2}+\frac{1}{4}i}$. This function $f(z)$ maps the unit disk $\mathbb{U}$ onto the following domain:



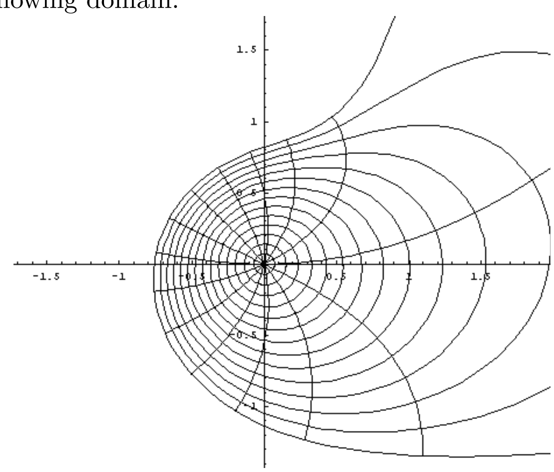
EXAMPLE 2. If we take $\alpha = \frac{2}{3} + \frac{1}{4}i$ in (5), then we have that

$$f(z) = \frac{z}{(1-z)^{\frac{184+69i}{438}}}.$$

This function $f(z)$ satisfies

$$\begin{aligned} \mathrm{Re}\left(\frac{1}{\alpha}\frac{zf'(z)}{f(z)}\right) &= \mathrm{Re}\left\{\frac{96-36i}{73}\left(1+\frac{(184+69i)z}{438(1-z)}\right)\right\} \\ &= \frac{96}{73} + \frac{46}{73}\mathrm{Re}\left(\frac{z}{1-z}\right) > \frac{96}{73} - \frac{23}{73} = 1. \end{aligned}$$

Thus we see that $f(z) \in \mathcal{S}_{\frac{2}{3}+\frac{1}{4}i}$. This function $f(z)$ maps the unit disk $\mathbb{U}$ onto the following domain:



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