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On Constructive Versions Of The Tychonoff And Schauder Fixed Point Theorems*

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Abstract

It is often demonstrated that Brouwer's fixed point theorem can not be constructively or computably proved. Therefore, Tychonoff's and Schauder's fixed point theorems also can not be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's theorem using Sperner's lemma. We present a constructive version of Tychonoff's fixed point theorem for a locally convex space using a constructive version of KKM (Knaster, Kuratowski and Mazurkiewicz) lemma, and a constructive version of Schauder's fixed point theorem for a Banach space as a corollary to that of Tychonoff's theorem. We follow the Bishop style constructive mathematics according to Bishop and Bridges [1], Bridges and Richman [2], Bridges and Vîţă [3] and Troelstra and Dalen [10].

1 Introduction

It is often demonstrated that Brouwer's fixed point theorem can not be constructively or computably proved (see Potgieter [6]). Indeterminacy of the intermediate value theorem is an example of non-constructivity of Brouwer's fixed point theorem. Therefore, Tychonoff's and Schauder's fixed point theorems also cannot be constructively proved. On the other hand, however, Sperner's lemma which is used to prove Brouwer's theorem can be constructively proved. Some authors have presented a constructive (or an approximate) version of Brouwer's theorem using Sperner's lemma. See Dalen [4] and Veldman [11].

We present a constructive version of Tychonoff's fixed point theorem for a locally convex space using a constructive version of KKM (Knaster, Kuratowski and Mazurkiewicz) lemma, and a constructive version of Schauder's fixed point theorem as a corollary to that of Tychonoff's theorem[1]. A constructive version of Tychonoff's fixed point theorem states that for any uniformly continuous function from a compact and convex subset of a locally convex space to itself there is an approximate fixed point, and that of Schauder's fixed point theorem states that for any uniformly continuous

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[1]Formulations of Tychonoff's and Schauder's fixed point theorems in this paper follow those in Istrăţescu [5].

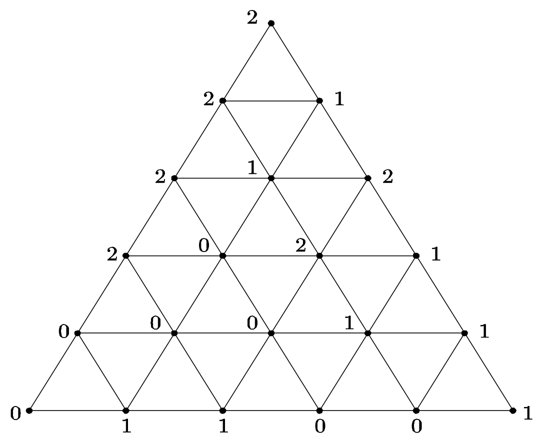

Figure 1: Partition and labeling of 2-dimensional simplex

function from a compact and convex subset of a Banach space to itself there is an approximate fixed point. An approximate fixed point x for a function f is a point which satisfies $|x - f(x)| < \varepsilon$ for any $\varepsilon > 0$ in terms of the norm in a Banach space, or satisfies $\sum_{i \in F} p_i(x - f(x)) < \varepsilon,\ i \in F$ for any $\varepsilon > 0$ in terms of each finite family F of seminorms in a locally convex space.

A Banach space is a locally convex space. Thus, Schauder's theorem is obtained as a corollary to Tychonoff's theorem. We follow the Bishop style constructive mathematics according to Bishop and Bridges [1], Bridges and Richman [2], Bridges and Vîţă [3] and Troelstra and Dalen [10].

2 Constructive Version of KKM Lemma

Let Δ denote an n-dimensional simplex. n is a positive integer at least 2. For example, a 2-dimensional simplex is a triangle. We partition the simplex. Let K denote the set of small n-dimensional simplices of Δ constructed by partition. The vertices of these small simplices of K are labeled with the numbers $0, 1, 2, ..., n$ subject to the following rules.

1. The vertices of Δ are respectively labeled with 0 to n. We label a point $(1, 0, \dots, 0)$ with 0, a point $(0, 1, 0, \dots, 0)$ with 1, a point $(0, 0, 1 \dots, 0)$ with 2, $\dots$, a point $(0, \dots, 0, 1)$ with n. That is, a vertex whose k-th coordinate ($k = 0, 1, \dots, n$) is 1 and all other coordinates are 0 is labeled with k.

2. If a vertex of K is contained in an $(n-1)$-dimensional face of Δ, then that vertex is labeled with some number which is the same as the number of a vertex of that face. It may be a vertex of the face of Δ or a vertex of a small simplex of K constructed by partition, which is contained in that face of Δ.

3. If a vertex of K is contained in an $(n-2)$-dimensional face of Δ, then that vertex is labeled with some number which is the same as the number of a vertex of that face. And so on for cases of higher dimension.

4. A vertex contained in inside of Δ is labeled with arbitrary number among $0, 1, 2, ..., n$.

A small simplex of K which is labeled with the numbers 0, 1, ..., n is called a *fully labeled simplex*.

Figure 1 is an example of partition and labeling of a simplex.

Then, we can get the following lemma.

LEMMA 1 (Sperner's lemma). If we label the vertices of K following above rules $1 \sim 4$, then there are an odd number of fully labeled simplices. Thus, there exists at least one fully labeled simplex.

For a constructive proof of this lemma see, for example, Su [7].

Now we prove the following lemma.

LEMMA 2 (Constructive version of KKM lemma). Let Δ be an n-dimensional simplex, $\Delta_i^k, k = 0, 1, \dots, n$ be k-dimensional faces of Δ, $\mathbf{p}^{i_0}, \mathbf{p}^{i_1}, \dots, \mathbf{p}^{i_k}$ be their vertices, and $A_0, A_1, \dots, A_n$ be inhabited subsets of Δ which satisfy the following condition[2].

$$\forall k\ \Delta_i^k \subset \bigcup_{j=0}^{k} A_{i_j}.$$

Then, for any $\varepsilon > 0$ we have $\bigcap_{i=0}^{n} V(A_i, \varepsilon) \neq \emptyset$, and we can find a point contained in $\bigcap_{i=0}^{n} V(A_i, \varepsilon)$, where $V(A_i, \varepsilon)$ is an ε-neighborhood of A_i.

The condition of this lemma means that, for example, when the vertices of Δ_i^3 are $\mathbf{p}^1, \mathbf{p}^4, \mathbf{p}^5$ and $\mathbf{p}^7$, Δ_i^3 is covered by A_1, A_4, A_5 and A_7. Similarly, when the vertices of Δ_i^4 are $\mathbf{p}^1, \mathbf{p}^4, \mathbf{p}^5, \mathbf{p}^6$ and $\mathbf{p}^7$, Δ_i^4 is covered by A_1, A_4, A_5, A_6 and A_7.

PROOF: Let K be the set of small n-dimensional simplices constructed by partition of an n-dimensional simplex Δ. The vertices $\mathbf{p}^0, \mathbf{p}^1, \dots, \mathbf{p}^n$ of Δ are labeled with, respectively, 0, 1, ... and n. Each vertex of all simplices of K is contained in some face of Δ including Δ itself. If a vertex $\mathbf{p}$ is contained in more than one faces of Δ, we select a face of least dimension. Denote it by Δ_i^k. By the assumption $\mathbf{p}$ is contained in at least one of $A_{i_0}, A_{i_1}, \dots$ and A_{i_k}. Denote it by A_{i_j}, and label p with i_j. By the condition of this lemma i_j is the number of one of the vertices of Δ_i^k. This labeling satisfies the conditions for Sperner's lemma. Thus, there exists a fully labeled n-dimensional simplex of K. Denote the vertices of this simplex by $\mathbf{q}^0, \mathbf{q}^1, \dots$ and $\mathbf{q}^n$. We can name them such that $\mathbf{q}^i$ is labeled with i. Then, each $\mathbf{q}^i$ is contained in A_i. If partition of Δ is sufficiently fine, the size of this fully labeled n-dimensional simplex is sufficiently small, and we can make all $V(A_i, \varepsilon)$'s contain this simplex. Then, this simplex is contained in the intersection of all $V(A_i, \varepsilon)$'s. Therefore, we have $\bigcap_{i=0}^{n} V(A_i, \varepsilon) \neq \emptyset$, and we can constructively find a point in this set.

[2]Usually in KKM lemma $A_0, A_1, \dots, A_n$ are assumed to be closed sets. But in this lemma we do not assume so.

3 Constructive Version of Tychonoff's Fixed Point Theorem

In this section we will prove a constructive version of Tychonoff's fixed point theorem using a constructive version of KKM lemma. The classical Tychonoff's theorem is stated as follows;

Tychonoff's fixed point theorem Let X be a compact and convex subset of a locally convex space E, and f be a continuous function from X to itself. Then, f has a fixed point.

A locally convex space consists of a vector space E and a family $(p_i)_{i\in I}$ of seminorms on X. I is an index set, for example, a set of positive integers. According to Bridges and Vîţă [3] uniform continuity of a function in a locally convex space is constructively defined as follows;

Uniform continuity of a function in a locally convex space Let X, Y be locally convex spaces. A function $f: X \longrightarrow Y$ is uniformly continuous on X if for each $\varepsilon > 0$ and each finitely enumerable subset G of J, which is also an index set, there exists $\delta > 0$ and a finitely enumerable subset F of I such that if $x, y \in X$ and $\sum_{i\in F} p_i(x-y) < \delta$, then $\sum_{j\in G} q_j(f(x)-f(y)) < \varepsilon$, where $(q_j)_{j\in J}$ is a family of seminorms on Y.

Also an approximate fixed point is defined as follows;

Approximate fixed point of a function in a locally convex space x is an approximate fixed point of a function f from X to itself if for any $\varepsilon > 0$ we have

$$\sum_{i\in F} p_i(x - f(x)) < \varepsilon$$

for each finitely enumerable $F \in I$.

According to Bridges and Vîţă [3] we define, constructively, total boundedness of a set in a locally convex space as follows.

Total boundedness of a set in a locally convex space Let X be a subset of E, F be a finitely enumerable subset of I, and $\varepsilon > 0$. By an ε-approximation to X relative to F we mean a subset T of X such that for each $x \in X$ there exists $y \in T$ with $\sum_{i\in F} p_i(x-y) < \varepsilon$.

X is totally bounded relative to F if for each $\varepsilon > 0$ there exists a finitely enumerable ε-approximation to X relative to F. It is totally bounded if it is totally bounded relative to each finitely enumerable subset of I.

The content of our constructive version of Tychonoff's fixed point theorem is described in the following theorem.

THEOREM 1 (Constructive version of Tychonoff's fixed point theorem). Let X be a compact, convex subset of a locally convex space E, and f be a uniformly continuous function from X to itself. Then, f has an approximate fixed point.

PROOF: We prove this theorem through two steps.

1. First we show the following result.

   Let X be a set in a locally convex space. To each $x \in X$, let a set $H(x)$ be given such that the convex hull of any finite subset $\{x_1, x_2, \dots, x_k\}$ of X is contained in $\bigcup_{j=1}^k H(x_j)$. Then, $\bigcap_{j=1}^m V(H(x_j), F, \varepsilon) \neq \emptyset$ for any finite positive integer m and each finitely enumerable $F \in I$. Where

   $$V(H(x), F, \varepsilon) = \{y \in X | \sum_{i \in F} p_i(y - z) < \varepsilon \text{ for some } z \in H(x)\}.$$

   It is called a basic neighborhood of $H(x)$.

   This is an extension to a locally convex space of a constructive version of KKM lemma (Lemma 2). Consider an $(n-1)$-dimensional simplex Δ in Euclidean space with vertices $v_1 = (1, 0, 0, \dots, 0)$, $v_2 = (0, 1, 0, \dots, 0)$, $\dots$, $v_m = (0, 0, \dots, 1)$. Denote a point $v \in \Delta$ as $v = \sum_{j=1}^m \alpha_j v_j$, and consider a function $g : \Delta \longrightarrow X$ by $g(v) = \sum_{j=1}^m \alpha_j x_j$, where $\sum_{j=1}^m \alpha_j = 1,\ \alpha_j \geq 0,\ j = 1, 2, \dots, m$. g is clearly a uniformly continuous function. $g(v_j) = x_j$ and $g^{-1}(x_j) = v_j$ for all j. Uniform continuity of g is described as follows;

   g is uniformly continuous on Δ if for each $\varepsilon > 0$ and each finitely enumerable subset F of I there exists $\delta > 0$ such that if $x, y \in \Delta$ and $|x - y| < \delta$, then $\sum_{i \in F} p_i(g(x) - g(y)) < \varepsilon$.

   Consider the following sets.

   $$K_j = g^{-1} H(x_j),\ j = 1, 2, \dots, m.$$

   For any indices $1 \leq i_1 < i_2 < \cdots < i_k \leq m$, the $(k-1)$-dimensional simplex $(v_{i_1}, v_{i_2}, \dots, v_{i_k})$ is contained in $K_{i_1} \cup K_{i_2} \cup \cdots \cup K_{i_k}$. By Lemma 2 this implies $\bigcap_{j=1}^m V(K_j, \delta) \neq \emptyset$ for any $\delta > 0$, where $V(K_i, \delta)$ is an δ-neighborhood of K_i. Because g is uniformly continuous, $\bigcap_{j=1}^m V(K_j, \delta) \neq \emptyset$ means $\bigcap_{j=1}^m V(H(x_j), F, \varepsilon) \neq \emptyset$ for any $\varepsilon > 0$ and each $F \in I$.

2. Since X is totally bounded, there exists a finitely enumerable η-approximation $\{x_1, x_2, \dots, x_m\}$ to X, that is, for each $x \in X$ we have $\sum_{i \in F} p_i(x - x_l) < \eta$ with any $\eta > 0$ for at least one $x_l,\ l = 1, 2, \dots, m$ for each $F \subset I$. Note that a point y^* is an approximate fixed point if for any $\varepsilon > 0$ we have

   $$\sum_{i \in F} p_i(x - f(x)) < \varepsilon.$$

   for each finitely enumerable $F \subset I$. Define a function Q from X to the set of all subsets of X by

   $$Q(x) = \{y \in X | \sum_{i \in F} p_i(y - f(y)) < \sum_{i \in F} p_i(x - f(y)) + \tau\},$$

   with $\tau > 0$. $x_{i_j},\ j = 1, 2, \dots, k$, is clearly contained in $Q(x_{i_j})$. Let $y = \sum_{j=1}^k \alpha_j x_{i_j}$ be a point in the convex hull of $\{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$, where $\sum_{j=1}^k \alpha_j =$

1, $\alpha_j \geq 0$. Then, since seminorms are subadditive[3], we have

$$\begin{aligned}\sum_{i\in F} p_i(y - f(y)) &= \sum_{i\in F} p_i\left(\sum_{j=1}^{k} \alpha_j(x_{i_j} - f(y))\right) \leq \sum_{i\in F}\sum_{j=1}^{k} \alpha_j p_i(x_{i_j} - f(y)) \\ &= \sum_{j=1}^{k} \alpha_j \sum_{i\in F} p_i(x_{i_j} - f(y)).\end{aligned}$$

This means that for at least one i_j we have

$$\sum_{i\in F} p_i(y - f(y)) < \sum_{i\in F} p_i(x_{i_j} - f(y)) + \tau.$$

Therefore, $y \in Q(x_{i_1}) \cup Q(x_{i_2}) \cup \cdots \cup Q(x_{i_k})$, and by (1) of this theorem

$$\bigcap_{j=1}^{m} V(Q(x_j), F, \varepsilon) \neq \emptyset$$

for any $\varepsilon > 0$. Let $y^* \in \bigcap_{j=1}^{m} V(Q(x_j), F, \varepsilon)$. Then, $\sum_{i\in F} p_i(y^* - \bar{y}) < \varepsilon$ for some $\bar{y}$ such that

$$\sum_{i\in F} p_i(\bar{y} - f(\bar{y})) < \sum_{i\in F} p_i(x_j - f(\bar{y})) + \tau \text{ for each } j.$$

Since ε may be arbitrarily small, uniform continuity of f implies

$$\sum_{i\in F} p_i(f(y^*) - f(\bar{y})) < \delta$$

for any $\delta > 0$. Thus,

$$\begin{aligned}\sum_{i\in F} p_i(y^* - f(y^*)) &\leq \sum_{i\in F}[p_i(y^* - \bar{y}) + p_i(\bar{y} - f(\bar{y})) + p_i(f(\bar{y}) - f(y^*))] \\ &< \sum_{i\in F} p_i(x_j - f(\bar{y})) + \delta + \tau + \varepsilon\end{aligned}$$

for each j. Since $f(y^*) \in X$ and X is totally bounded, $\sum_{i\in F} p_i(x_j - f(y^*)) < \eta$ for at least one x_j. Therefore,

$$\begin{aligned}\sum_{i\in F} p_i(y^* - f(y^*)) &< \sum_{i\in F}[p_i(x_j - f(y^*)) + p_i(f(y^*) - f(\bar{y}))] + \delta + \tau + \varepsilon \\ &< \eta + 2\delta + \tau + \varepsilon.\end{aligned}$$

Since $\eta + 2\delta + \tau + \varepsilon$ may be arbitrarily small, y^* is an approximate fixed point of f.

[3] Subadditivity of a seminorm p_i means that for $x, y \in X$ $p_i(x + y) \leq p_i(x) + p_i(y)$.

A Banach space is a locally convex space. Therefore, as a corollary to the constructive version of Tychonoff's fixed point theorem we obtain the following theorem.

THEOREM 2 (Constructive version of Schauder's fixed point theorem). Let X be a compact (totally bounded and complete) and convex subset of a Banach space E, and f be a uniformly continuous function from X to itself. Then, f has an approximate fixed point.

x is an approximate fixed point of f if for any $\varepsilon > 0$

$$|x - f(x)| < \varepsilon$$

in terms of the norm in a Banach space.

4 Concluding Remarks

In other papers we studied some problems in economic theory and game theory from the viewpoint of constructive mathematics as follows.

1. In Tanaka [8] we have proved that the existence of an approximate Nash equilibrium in a strategic game is derived from a constructive version of Brouwer's fixed point theorem.
2. In Tanaka [9] we have proved that the existence of an approximate equilibrium in a competitive exchange economy with single-valued excess demand functions is derived from a constructive version of Brouwer's fixed point theorem. Also we have shown that the so-called Uzawa equivalence theorem, which states that the existence of an equilibrium in a competitive exchange economy and Brouwer's fixed point theorem are equivalent, approximately holds.

The results of this paper are extensions of the constructive version of Brouwer's fixed point theorem used in these studies to a Banach space and a locally convex space through a constructive version of KKM lemma.

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