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Oscillation Criteria For Second Order Delay Differential Equations With Mixed Nonlinearities*

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Abstract

In this paper we establish oscillation criteria for second order delay differential equations with mixed nonlinearlities. The results obtained here generalize some of the existing results.

1 Introduction

Consider a second order delay differential equation of the form

$$(r(t)|x'^{\alpha-1}x'(t))' + q(t)|x(\tau_0(t))|^{\alpha-1}x(\tau_0(t)) + \sum_{j=1}^{n} q_j(t)|x(\tau_j(t))|^{\alpha_j-1}x(\tau_j(t)) = 0 \quad (1)$$

where $\alpha_1 > ... > \alpha_m > \alpha > \alpha_{m+1} > \cdots \alpha_n > 0$, $n > m \geq 1$, are constants, $r(t) \in C^1[t_0,\infty), r(t) > 0, q(t)$ and $q_j(t) \in C[t_0,\infty), j = 1,2,...,n$, are nonnegative. Here we assume that there exists $\tau(t) \in C^1[t_0,\infty)$ such that $\tau(t) \leq \tau_j(t), \tau(t) \leq t, \lim_{t\longrightarrow\infty} \tau(t) = \infty$ and $\tau^{'}(t) \geq 0$ for $t \in [t_0,\infty), j = 0,1,2,...,n$.

By a solution of equation (1), we mean a function $x \in C^1[T_x,\infty), T_x \geq t_0$, which has the property $r(t)|x'^{\alpha-1}x'^1[T_x,\infty)$ and satisfies the equation for all $t \geq T_x$. We restrict our attention to those solutions $x(t)$ of equation (1) which satisfy $\sup\{|x(t)| : t > T\} > 0$ for all $T \geq T_x$. Such a solution is said to be oscillatory if it has a sequence of zeros tending to infinity and nonoscillatory otherwise.

Particular cases of equation (1) has been considered in [1, 2, 4, 5] and they established conditions for the oscillation of all solutions under the assumption

$$\lim_{t\longrightarrow\infty} R(t) = \infty, \text{ where } R(t) = \int_{t_0}^{t} \frac{1}{r^{\frac{1}{\alpha}}(s)} ds. \quad (2)$$

In this paper, we shall further investigate and extend the main results in [4] and [5] to the general equation (1) with mixed nonlinearities and several delays since such type of equation arise in the growth of bacteria population with competitive species.

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2 Main Results

We first present a lemma which is a generalization of Lemma 1 of Sun and Wong [6].

LEMMA 1. Let $\{\alpha_i\}, i = 1, 2, ..., n$, be the n-tuple satisfying $\alpha_1 > \cdots > \alpha_m > \alpha > \alpha_{m+1} > \cdots \alpha_n > 0$. Then there is an n-tuple $(\eta_1, \eta_2, ..., \eta_n)$ satisfying

$$\sum_{i=1}^{n} \alpha_i \eta_i = \alpha,$$

and

$$\sum_{i=1}^{n} \eta_i = 1, \quad 0 < \eta_i < 1.$$

LEMMA 2. Suppose X and Y are nonnegative. Then

$$X^\gamma - \gamma X Y^{\gamma-1} + (\gamma - 1) Y^\gamma \geq 0, \gamma > 1$$

where equality holds if and only if $X = Y$.

The proof of the lemma can be found in [3].

THEOREM 1. Assume that (2) holds and

$$\int^{\infty} \Big(R^\alpha(\tau(t)) Q(t) - (\frac{\alpha}{\alpha+1})^{\alpha+1} \frac{\tau^{'}(t)}{R(\tau(t)) r^{\frac{1}{\alpha}}(\tau(t))} \Big) dt = \infty \tag{3}$$

where

$$Q(t) = q(t) + k \prod_{i=1}^{n} q_i^{\eta_i}(t),\ k = \prod_{i=1}^{n} \eta_i^{-\eta_i}$$

and $\eta_1, \eta_2, ..., \eta_n$ are positive constants as in Lemma 1. Then every solution of equation (1) is oscillatory.

PROOF . Suppose that $x(t)$ is a nonoscillatory solution of equation (1). Without loss of generality we may assume that $x(t) > 0$ for all large t since the case $x(t) < 0$ can be considered by the same method. From equation (1) and condition (2) we can easily obtain that there exists a $t_1 > t_0$ such that $x(t) > 0, x^{'}(t) > 0, (r(t)(x^{'}(t))^\alpha)^{'} \leq 0, t \geq t_1$. Therefore, we have that

$$r(t)(x^{'}(t))^\alpha \leq (r(\tau(t))(x^{'}(\tau(t)))^\alpha$$

for $t \geq t_1$ which implies that

$$\frac{x^{'}(\tau(t))}{x^{'}(t)} \geq \Big(\frac{r(t)}{r(\tau(t))} \Big)^{\frac{1}{\alpha}} \text{for } t \geq t_1. \tag{4}$$

Define

$$W(t) = R^\alpha(\tau(t)) \frac{r(t) x^{'}(t)^\alpha}{x(\tau(t))^\alpha} \text{ for } t \geq t_1. \tag{5}$$

Then $W(t) > 0$. From equations (1) and (5) and noting that $x'(t) > 0$ and hence $x(\tau_j(t)) \geq x(\tau(t))$ for $j = 0, 1, 2, ..., n$, we have

$$\begin{aligned} W^{'}(t) \quad &\leq \quad \frac{\alpha\tau^{'}(t)R^{\alpha-1}(\tau(t))}{r^{\frac{1}{\alpha}}(\tau(t))}\frac{r(t)(x^{'}(t))^{\alpha}}{(x(\tau(t)))^{\alpha}} - R^{\alpha}(\tau(t))q(t) \\ &\quad -\alpha R^{\alpha}(\tau(t))\frac{r(t)(x^{'}(t))^{\alpha}}{x^{\alpha+1}(\tau(t))}x^{'}(\tau(t))\tau^{'}(t) - R^{\alpha}(\tau(t))\sum_{j=1}^{n} q_j(t)x^{\alpha_j-\alpha}(\tau(t)). \end{aligned} \tag{6}$$

Recall the arithmetic-geometric inequality

$$\sum_{i=1}^{n}\eta_i u_i \geq \prod_{i=1}^{n} u_i^{\eta_i}, u_i \geq 0 \tag{7}$$

where $\eta_1, ..., \eta_n$ are chosen according to given $\alpha, \alpha_1 ..., \alpha_n$ as in Lemma 1. Now return to (6) and identify $u_i = \eta_i^{-1}q_i(t)x^{\alpha_i-\alpha}(\tau(t))$ in (7) to obtain

$$\begin{aligned} W^{'}(t) \quad &\leq \quad -R^{\alpha}(\tau(t))Q(t) + \frac{\alpha\tau^{'}(t)}{R(\tau(t))r^{\frac{1}{\alpha}}(\tau(t))}W(t) \\ &\quad -\frac{\alpha\tau^{'}(t)}{R(\tau(t))r^{\frac{1}{\alpha}}(\tau(t))}\frac{R^{\alpha+1}(\tau(t))r^{\frac{\alpha+1}{\alpha}}(t)(x^{'}(t))^{\alpha+1}}{(x(\tau(t)))^{\alpha+1}} \\ &= \quad -R^{\alpha}(\tau(t))Q(t) + \frac{\alpha\tau^{'}(t)}{R(\tau(t))r^{\frac{1}{\alpha}}(\tau(t))}[W(t) - W^{\frac{\alpha+1}{\alpha}}(t)] \end{aligned} \tag{8}$$

where $Q(t)$ is the same as in Theorem 1. Set $X = W(t)$ and $Y = \lambda^{\frac{1}{1-\lambda}}$ where $\lambda = \frac{\alpha+1}{\alpha} > 1$. Applying Lemma 2 in (8) we obtain

$$W^{'}(t) \leq \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\tau^{'}(t)}{R(\tau(t))r^{\frac{1}{\alpha}}(\tau(t))} - R^{\alpha}(\tau(t))Q(t).$$

Integrating the last inequality from t_1 to t, we have

$$0 < W(t) \leq W(t_1) - \int_{t_1}^{t}(R^{\alpha}(\tau(s))Q(s) - \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}\frac{\tau^{'}(s)}{R(\tau(s))r^{\frac{1}{\alpha}}(\tau(s))})ds. \tag{9}$$

Letting $t \to \infty$ in (9), we obtain a contradiction with (3). This completes the proof.

Based on Theorem 1 and the proofs of Corollary 2.1 and the Corollary 2.2 in [2, 5], we can easily obtain the following results.

COROLLARY 2. Assume that (2) holds and for $t_1 > t_0$

$$\lim_{t\to\infty}\inf\frac{1}{\log R(\tau(t))}\int_{t_1}^{t}R^{\alpha}(\tau(s))Q(s)ds > \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}$$

where $Q(t)$ is the same as in Theorem 1. Then every solution of equation (1) is oscillatory.

COROLLARY 3. Assume that (2) holds, $\tau^{'}(t) > 0$ and

$$\liminf_{t\to\infty} \frac{R^{\alpha+1}(\tau(t))r^{\frac{1}{\alpha}}(\tau(t))}{\tau^{'}(t)}Q(t) > \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha+1}$$

where $Q(t)$ is the same as in Theorem 1. Then every solution of equation (1) is oscillatory.

The following examples show the importance of our main results.

EXAMPLE 1. Consider the equation

$$((x^{'}(t))^{\frac{3}{5}})^{'} + \frac{a}{t^{\frac{8}{5}}}x^{\frac{3}{5}}(\lambda_1 t) + \frac{b}{t^4}x^{\frac{5}{3}}(\lambda_2 t) + \frac{c}{t}x^{\frac{1}{3}}(\lambda_3 t) = 0, \quad t \geq 1 \tag{10}$$

where $0 < \lambda_i < 1$ for $i = 1, 2, 3$ and $a, b, c > 0$ are constants. Set $\tau(t) = \lambda t$ with $\lambda = \min\{\lambda_1, \lambda_2, \lambda_3\}$. Also $\alpha = 3/5, \alpha_1 = 5/3, \alpha_2 = 1/3$. By direct computation, we have by choosing $\eta_1 = \frac{1}{5}, \eta_2 = \frac{4}{5}$, that

$$Q(t) = \frac{\Big(a + 5(\frac{1}{4})^{4/5}\sqrt[5]{bc^4}\Big)}{t^{8/5}}.$$

By Corollary 2 or Corollary 3 we have that all solutions of equation (10) are oscillatory if

$$\lambda^{3/5}\Big(a + 5(\frac{1}{4})^{4/5}\sqrt[5]{bc^4}\Big) > \Big(\frac{3}{8}\Big)^{8/5}.$$

EXAMPLE 2. Consider the equation

$$x^{''}(t) + \frac{a}{t^2}x(\lambda_1 t) + \frac{b}{t^3}x^{\frac{7}{3}}(\lambda_2 t) + \frac{c}{t^2}x^{\frac{5}{3}}(\lambda_3 t) + \frac{d}{t^{\frac{12}{7}}}x^{\frac{1}{3}}(\lambda_4 t) = 0, \quad t \geq 1 \tag{11}$$

where $0 < \lambda_i < 1$ for $i = 1, 2, 3, 4$ and $a, b, c, d > 0$ are constants. Set $\tau(t) = \lambda t$ with $\lambda = \min\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$. Also $\alpha = 1, \alpha_1 = 7/3, \alpha_2 = 5/3, \alpha_3 = 1/3$. By direct computation, we have by choosing $\eta_1 = 1/6, \eta_2 = 1/4, \eta_3 = 7/12$, that

$$Q(t) = \frac{\Big(a + kb^{\frac{1}{6}}c^{\frac{1}{4}}d^{\frac{7}{12}}\Big)}{t^2}, \quad k = \frac{2^{\frac{11}{6}}3^{\frac{3}{4}}}{7^{\frac{7}{12}}}.$$

By Corollary 2 or Corollary 3 we have that all solutions of equation (11) are oscillatory if

$$\lambda\Big(a + kb^{\frac{1}{6}}c^{\frac{1}{4}}d^{\frac{7}{12}}\Big) > \frac{1}{4}.$$

3 Remark

The main results of this paper can be easily extended to the following neutral differential equation.

$$(r(t)|z'^{\alpha-1}z'(t))'^{\alpha-1}x(\tau(t)) + \sum_{j=1}^{n} q_j(t)|x'(\tau_j(t))|^{\alpha_j-1}x(\tau_j(t)) = 0$$

where $z(t) = x(t)+p(t)x(t-\sigma)$ with $0 \leq p(t) < 1$and $\sigma \geq 0$ and the details are skipped.

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