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On The Defining Number For Vertex Colorings Of A Family Of Graphs*

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Abstract

In a given graph $G = (V, E)$ a set S of vertices with an assignment of colors to them is called a defining set for vertex colorings of G if there exists a unique extension of S to a $c \geq \chi(G)$ coloring of the vertices of G. A defining set with minimum cardinality is called a minimum defining set, and its cardinality is the defining number. In this paper, we study the defining number for vertex colorings of graphs arising from applying Mycielski's construction to Ladder graphs.

1 Introduction

A *c-coloring* of a graph G is an assignment of c different colors to the vertices of G such that adjacent vertices receive different colors. The (vertex) *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number c for which there exists a c-coloring of G. A graph G with $\chi(G) = c$ is called a *c-chromatic graph*, (see [7]).

For a graph G and a number $c \geq \chi(G)$, a subset of vertices S with an assignment of colors to them is called a *defining set for vertex colorings* if there exists a unique extension of the colors of S to a c-coloring of the vertices of G. A defining set with minimum cardinality is called a *minimum defining set* and its cardinality is the *defining number*, denoted by $d(G, c)$.

The concept of defining sets has been studied to some extent, for block designs and under another name, *critical sets* for Latin squares, and *forcing sets* for perfect matchings in graphs, also in dominating sets, geodetic sets, and hull sets in graphs, (see [1–6]).

The concept of defining set for vertex colorings is closely related to the concept of *list coloring*. In a list coloring for each vertex v there is a given list of colors $L(v)$ available on that vertex. Any defining set S in a graph G naturally induces a list of possible colors for the vertices of the induced subgraph $\langle G - S\rangle$. Furthermore, using this list of colors, $\langle G - S\rangle$ is uniquely list colorable, (see [1,3]).

A graph G with n vertices is called a uniquely 2-list colorable graph if there exists a list $L(v)$ of at least two colors for each $v \in V(G)$ such that G has a unique list coloring with respect to these lists.

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Mahdian et al. in [3] obtained the following characterization.

Theorem A ([3]). A connected graph is uniquely 2-list colorable if and only if at least one of its blocks is not a cycle, a complete graph or a complete bipartite graph.

For a simple graph G, by graph $M(G)$ we mean the graph arising from applying *Mycielski's construction* to G. Mycielski's construction produces a simple graph $M(G)$ containing G as follows. If $V(G) = \{v_1, v_2, \ldots, v_n\}$, then $V(M(G)) = \{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n, w\}$, where $V(G) \cap \{u_1, u_2, \ldots, u_n, w\} = \emptyset$ and

$$E(M(G)) = E(G) \cup \{u_i v \mid v \in N_G(v_i),\ 1 \leq i \leq n\} \cup \{u_i w \mid 1 \leq i \leq n\}.$$

Theorem B ([7]). If G is a c-chromatic triangle-free graph, then $M(G)$ is a $(c+1)$-chromatic triangle-free graph.

The *cartesian product* of two graphs G and H written $G \times H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting (u, v) adjacent to (u', v') if and only if either (1) $u = u'$ and $vv' \in E(H)$, or (2) $v = v'$ and $uu' \in E(G)$.

Mojdeh et al. in [6] obtained the defining number for vertex colorings in graphs obtained from paths, cycles, complete graphs and complete bipartite graphs applying Mycielski's construction. For $n \geq 2$, we refer $K_2 \times P_n$ as the *Ladder graph* of order $2n$. Here we investigate the defining number for vertex colorings of the graph G arising from applying Mycielski's construction to the Ladder graph $K_2 \times P_n$ for any integer $n \geq 2$. In this paper for a vertex v, $c(v)$ denotes the color of v.

2 Main Results

It follows from Theorem B that for any positive integer n, $\chi(M(K_2 \times P_n)) = 3$. In this section, we present our main results. We investigate $d(M(K_2 \times P_n), 3)$ for any positive integer n. For this purpose we denote the vertex set of the Ladder graph $G = K_2 \times P_n$ by

$$V = V(G) = \{a_{01}, a_{02}, ..., a_{0n}, a_{11}, a_{12}, ..., a_{1n}\},$$

where a_{ij} is adjacent to $a_{i(j-1)}, a_{i(j+1)}, a_{(i+1)j}$ for all $1 \leq j \leq n$, and $i+1$ is calculated in modulo 2. Then $V(M(G)) = V \cup U \cup \{w\}$, where

$$U = \{b_{01}, b_{02}, ..., b_{0n}, b_{11}, b_{12}, ..., b_{1n}\},$$

and $N_{M(G)}(b_{ij}) = N_G(a_{ij}) \cup \{w\}$ for all $i \in \{0, 1\}$ and $1 \leq j \leq n$. Figure 1 shows the graph $M(K_2 \times P_3)$ in which all of non-filled circles refer to the vertex w.

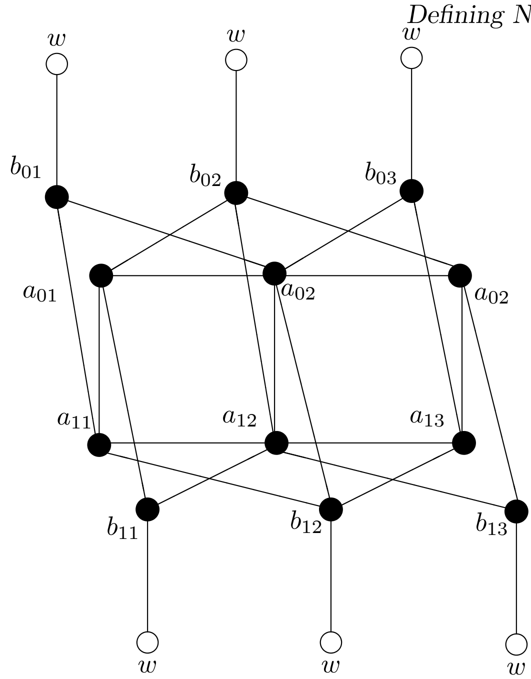

Figure 1. $M(K_2 \times P_3)$.

It follows from Theorem A that any defining set for vertex coloring of $M(K_2 \times P_n)$ intersects both $\{a_{01}, a_{11}, b_{01}, b_{11}\}$ and $\{a_{0n}, a_{1n}, b_{0n}, b_{1n}\}$. So we have the following lemma.

LEMMA 1. If S is a minimum defining set for vertex coloring of $M(K_2 \times P_n)$ for some positive integer n, then $S \cap \{a_{01}, a_{11}, b_{01}, b_{11}\} \neq \emptyset$ and $S \cap \{a_{0n}, a_{1n}, b_{0n}, b_{1n}\} \neq \emptyset$.

We next obtain the exact value of the defining number for $2 \leq n \leq 6$.

THEOREM 2. $d(M(K_2 \times P_n), \chi) = \begin{cases} 3 & 2 \leq n \leq 5 \\ 4 & n = 6 \end{cases}$.

PROOF. It follows from Lemma 1 that $d(M(K_2 \times P_n), \chi) \geq 2$, for $n = 2, 3, 4, 5, 6$. If S is a set of two vertices with colors in $M(K_2 \times P_n)$ such that

$$|S \cap \{a_{01}, a_{11}, b_{01}, b_{11}\}| = |S \cap \{a_{0n}, a_{1n}, b_{0n}, b_{1n}\}| = 1,$$

then by a case-checking we can examine all possibilities for S and notice that S is not a defining set for a 3-coloring of $M(K_2 \times P_n)$. In each possibility for the vertices of S we notice that either there was no extension of S to a 3-coloring of $M(G)$, or there were two different colorings containing S. So, $d(M(K_2 \times P_n), \chi) \geq 3$ for $n = 2, 3, 4, 5, 6$. Furthermore, no three vertices with colors uniquely determine a 3-coloring of $M(K_2 \times P_6)$. So $d(M(K_2 \times P_6), \chi) \geq 4$. Now we define the defining set S_n for $2 \leq n \leq 6$ as follows.

- For $n = 2$ we define $S_2 = \{w, a_{01}, a_{12}\}$, where $c(w) = 1$, $c(a_{01}) = 2$ and $c(a_{12}) = 1$.
- For $n = 3$ we define $S_3 = \{b_{01}, b_{03}, a_{13}\}$, where $c(b_{01}) = 2$, $c(b_{03}) = 3$ and $c(a_{13}) = 2$.
- For $n = 4$ we define $S_4 = \{b_{01}, b_{03}, a_{14}\}$, where $c(b_{01}) = 2$, $c(b_{03}) = 3$ and $c(a_{14}) = 1$.
- For $n = 5$ we define $S_5 = \{b_{05}, b_{11}, a_{13}\}$, where $c(b_{05}) = 2$, $c(b_{11}) = 3$ and $c(a_{13}) = 2$.
- For $n = 6$ we define $S_6 = \{b_{05}, b_{11}, a_{13}, b_{16}\}$, where $c(b_{05}) = 2$, $c(b_{11}) = 3$, $c(a_{13}) = 2$ and $c(b_{16}) = 3$.

Then S_n is a defining set for a 3-coloring of $M(K_2 \times P_n)$ for each $n = 2, 3, 4, 5, 6$. The proof is complete.

We proceed with the following lemma. For a set of vertices $X = \{v_1, v_2, ..., v_k\}$ and a coloring c we denote $c(X) = \{c(v_1), c(v_2), ..., c(v_k)\}$. Also if $X' = \{v'_1, v'_2, ..., v'_k\}$, then $c(X) = c(X')$ if and only if $c(v_i) = c(v'_i)$ for all $i = 1, 2, ..., k$.

LEMMA 3. If S is a minimum defining set for coloring of $M(K_2 \times P_n)$ for $n \geq 6$, then $S \cap \{a_{0(i+j)}, a_{1(i+j)}, b_{0(i+j)}, b_{1(i+j)} : 0 \leq j \leq 3\} \neq \emptyset$ for each $i = 1, 2, ..., n-3$.

PROOF. Let $n \geq 6$ and let S be a minimum defining set for a 3-coloring of $M(K_2 \times P_n)$. Assume to the contrary that $S \cap \{a_{0(i+j)}, a_{1(i+j)}, b_{0(i+j)}, b_{1(i+j)} : 0 \leq j \leq 3\} = \emptyset$ for some integer $i \in \{1, 2, ..., n-3\}$. Without loss of generality we may assume that $c(w) = 1$ in this 3-coloring. We checked all of the different colors for the vertices of

$$X = \{a_{0(i-1)}, b_{0(i-1)}, a_{1(i-1)}, b_{1(i-1)}, a_{0(i+4)}, b_{0(i+4)}, a_{1(i+4)}, b_{1(i+4)}\}$$

and noticed that in each case either there was no extension of S to a 3-coloring of $M(K_2 \times P_n)$ or there were at least two different 3-colorings containing S. We leave giving all cases and only present two samples. The other cases are similarly verified.

- If the coloring of X was given by $c(a_{0(i-1)}) = 1$, $c(b_{0(i-1)}) = 2$, $c(a_{1(i-1)}) = 3$, $c(b_{1(i-1)}) = 2$, $c(a_{0(i+4)}) = 1$, $c(b_{0(i+4)}) = 3$, $c(a_{1(i+4)}) = 2$, $c(b_{1(i+4)}) = 3$, then there were two colorings for the vertices of $A = \{a_{0(i+j)}, a_{1(i+j)}, b_{0(i+j)}, b_{1(i+j)} : 0 \leq j \leq 3\}$ as follows:

  (1) $c(a_{0i}) = 3$, $c(b_{0i}) = 2$, $c(a_{0(i+1)}) = 1$, $c(b_{0(i+1)}) = 2$, $c(a_{0(i+2)}) = 3$, $c(b_{0(i+2)}) = 3$, $c(a_{0(i+3)}) = 2$, $c(b_{0(i+3)}) = 2$, $c(a_{1i}) = 1$, $c(b_{1i}) = 2$, $c(a_{1(i+1)}) = 3$, $c(b_{1(i+1)}) = 3$, $c(a_{1(i+2)}) = 2$, $c(b_{1(i+2)}) = 2$, $c(a_{1(i+3)}) = 1$, $c(b_{1(i+3)}) = 3$, and

  (2) $c(a_{0i}) = 3$, $c(b_{0i}) = 3$, $c(a_{0(i+1)}) = 1$, $c(b_{0(i+1)}) = 2$, $c(a_{0(i+2)}) = 3$, $c(b_{0(i+2)}) = 3$, $c(a_{0(i+3)}) = 2$, $c(b_{0(i+3)}) = 2$, $c(a_{1i}) = 1$, $c(b_{1i}) = 2$, $c(a_{1(i+1)}) = 3$, $c(b_{1(i+1)}) = 3$, $c(a_{1(i+2)}) = 2$, $c(b_{1(i+2)}) = 2$, $c(a_{1(i+3)}) = 1$, $c(b_{1(i+3)}) = 3$.

- If the coloring of X was given by $c(a_{0(i-1)}) = 1$, $c(b_{0(i-1)}) = 2$, $c(a_{1(i-1)}) = 3$, $c(b_{1(i-1)}) = 2$, $c(a_{0(i+4)}) = 1$, $c(b_{0(i+4)}) = 3$, $c(a_{1(i+4)}) = 2$, $c(b_{1(i+4)}) = 3$, then $b_{1(i+3)}$ was a non-colorable vertex.

This contradiction completes the proof.

In the following we determine the exact value for $d(M(K_2 \times P_n), \chi)$ when $n \equiv 2$ or 3 $(mod\ 4)$.

THEOREM 4. For each $n \geq 6$ with $n \equiv 2$ or 3 $(mod\ 4)$, $d(M(K_2 \times P_n), \chi) = 2 + \left\lfloor \frac{n-2}{4} \right\rfloor$.

PROOF. Let $n \geq 6$ and $n \equiv 2$ or 3 $(mod\ 4)$. It follows from Lemmas 1 and 3 that $d(M(K_2 \times P_n), \chi) \geq 2 + \left\lfloor \frac{n-2}{4} \right\rfloor$. Now it is sufficient to define a defining set of required cardinality.

- For $n \equiv 2 \pmod 8$ we define $S = \{a_{11}, b_{0(8i+3)}, b_{1(8j-1)}, a_{0n} : i \geq 0, j \geq 1\}$, where $c(a_{11}) = 2$, $c(b_{0(8k+3)}) = 2$, $c(b_{1(8k-1)}) = 3$ and $c(a_{0n}) = 1$.
- For $n \equiv 0, 1 \pmod 8$ we define $S = \{a_{11}, b_{0(8i+3)}, b_{1(8j-1)}, a_{1n} : i \geq 0, j \geq 1\}$, where $c(a_{11}) = 2$, $c(b_{0(8k+3)}) = 2$, $c(b_{1(8k-1)}) = 3$ and $c(a_{1n}) = 1$.

Then S is a defining set of cardinality $2 + \left\lfloor \frac{n-2}{4} \right\rfloor$. Hence $d(M(K_2 \times P_n), \chi) \leq 2 + \left\lfloor \frac{n-2}{4} \right\rfloor$. Hence the the proof is complete.

Henceforth we let $n \geq 8$ and $n \equiv 0$ or 1 $(mod\ 4)$.

THEOREM 5. For $n \in \{8, 9\}$, $d(M(K_2 \times P_n), \chi) = 4$.

PROOF. It follows from Lemmas 1 and 3 that $d(M(K_2 \times P_n), \chi) \geq 3$ for $n \in \{8, 9\}$. Also it follows from Lemma 1 that for any defining set S,

$$S \cap \{a_{01}, a_{11}, b_{01}, b_{11}\} \neq \emptyset, S \cap \{a_{0n}, a_{1n}, b_{0n}, b_{1n}\} \neq \emptyset.$$

If S is a defining set for $M(K_2 \times P_8)$ and $|S| = 3$, then

$$|S \cap \{a_{04}, a_{14}, b_{04}, b_{14}, a_{05}, a_{15}, b_{05}, b_{15}\}| = 1.$$

But we examined all possibilities for S and noticed that either there were two colorings containing S, or there was a non-colorable vertex in $M(K_2 \times P_8)$. This implies that $d(M(K_2 \times P_8), \chi) \geq 4$. Similarly $d(M(K_2 \times P_9), \chi) \geq 4$. On the other hand we consider the following defining sets.

- For $n = 8$ we define $S = \{a_{11}, b_{03}, b_{17}, b_{08}\}$, where $c(a_{11}) = 2$, $c(b_{03}) = 2$, $c(b_{17}) = 3$ and $c(b_{08}) = 3$.
- For $n = 9$ we define $S' = \{a_{11}, b_{03}, b_{17}, a_{09}\}$, where $c(a_{11}) = 2$, $c(b_{03}) = 2$, $c(b_{17}) = 3$ and $c(a_{09}) = 3$.

Then S is a defining set for a 3-coloring of $M(K_2 \times P_8)$ and S' is a defining set for a 3-coloring of $M(K_2 \times P_9)$. Hence, the the proof is complete.

LEMMA 6. Let $n \geq 12$ and $n \equiv 0$ or 1 $(mod\ 4)$ and let S be a defining set for coloring of $M(K_2 \times P_n), \chi)$. If there is an integer $k > 1$ such that $|S \cap \{a_{0k}, a_{1k}, b_{0k}, b_{1k}\}| = \left|S \cap \{a_{0(k+4)}, a_{1(k+4)}, b_{0(k+4)}, b_{1(k+4)}\}\right| = 1$ and $|S \cap \{a_{0j}, a_{1j}, b_{0j}, b_{1j}\}| = 0$ for any j where $k < j < k+4$, then either $\{b_{0k}, b_{1(k+4)}\} \subseteq S$ where $c(b_{0k}) \neq c(b_{1(k+4)})$ or

$\{b_{1k}, b_{0(k+4)}\} \subseteq S$ where $c(b_{1k}) \neq c(b_{0(k+4)})$. Furthermore in the unique coloring containing S, $(c(b_{0k}), c(a_{0k})) = (c(a_{1(k+4)}), c(b_{1(k+4)}))$ and $(c(b_{1k}), c(a_{1k})) = (c(a_{0(k+4)}), c(b_{0(k+4)}))$.

PROOF. Let $n \geq 12$ and $n \equiv 0$ or 1 $(mod\ 4)$, and let S be a defining set for coloring of $M(K_2 \times P_n), \chi)$. Let $k > 1$ be a positive integer and

$$|S \cap \{a_{0k}, a_{1k}, b_{0k}, b_{1k}\}| = \left|S \cap \{a_{0(k+4)}, a_{1(k+4)}, b_{0(k+4)}, b_{1(k+4)}\}\right| = 1,$$

and $|S \cap \{a_{0j}, a_{1j}, b_{0j}, b_{1j}\}| = 0$ for any j where $k < j < k+4$. We consider the induced subgraph

$$G' = G[\{a_{ij}, b_{ij}, w : k-1 \leq j \leq k+5, 0 \leq i \leq 1\}].$$

Without loss of generality assume that $\{w, a_{ij}, b_{ij} : 0 \leq i \leq 1, j \in \{k-1, k+5\}\}$ has been colored and $c(w) = 1$. We consider the following cases.

- $\{b_{0k}, b_{1(k+4)}\} \nsubseteq S$ and $\{b_{1k}, b_{0(k+4)}\} \nsubseteq S$.
- $\{b_{0k}, b_{1(k+4)}\} \subseteq S$ and $c(b_{0k}) = c(b_{1(k+4)})$.
- $\{b_{1k}, b_{0(k+4)}\} \subseteq S$ and $c(b_{1k}) = c(b_{0(k+4)})$.

In each case we checked all possibilities for $S \cap V(G')$ and noticed that in each possibility either there were two colorings of G' containing

$$(S \cap V(G')) \cup \{w, a_{ij}, b_{ij} : 0 \leq i \leq 1, j \in \{k-1, k+5\}\}$$

or there was a non-colorable vertex in G'. So either $\{b_{0k}, b_{1(k+4)}\} \subseteq S$ where $c(b_{0k}) \neq c(b_{1(k+4)})$, or $\{b_{1k}, b_{0(k+4)}\} \subseteq S$ where $c(b_{1k}) \neq c(b_{0(k+4)})$. Also if $\{b_{0k}, b_{1(k+4)}\} \subseteq S$, or $\{b_{1k}, b_{0(k+4)}\} \subseteq S$, then $(c(b_{0k}), c(a_{0k})) = (c(a_{1(k+4)}), c(b_{1(k+4)}))$ and $(c(b_{1k}), c(a_{1k})) = (c(a_{0(k+4)}), c(b_{0(k+4)}))$. Hence, the the proof is complete.

THEOREM 7. For each $n \geq 8$ with $n \equiv 0$ or 1 $(mod\ 4)$, $d(M(K_2 \times P_n), \chi) = 2 + \left\lceil \frac{n-2}{4} \right\rceil$.

PROOF. Let $n \geq 8$ and $n \equiv 0$ or 1 $(mod\ 4)$. It follows from Lemmas 1 and 3 that $d(M(K_2 \times P_n), \chi) \geq 2 + \left\lfloor \frac{n-2}{4} \right\rfloor$. Let S be a defining set for coloring of $M(K_2 \times P_n)$ and let $c(w) = 1$ in the unique coloring c containing S. We proceed with Claim 1.

CLAIM 1. $|S| \geq 2 + \left\lceil \frac{n-2}{4} \right\rceil$.

To see the proof of our claim, we employ induction on n. For $n = 8, 9$ the result follows from Theorem 5. So assume that the result holds for any positive integer less than n. Let $n \geq 12$. It follows from Lemmas 1, 3 and 6 that there is a positive integer $k > 1$ such that

$$\begin{aligned} |S \cap \{a_{0k}, a_{1k}, b_{0k}, b_{1k}\}| &= \left|S \cap \{a_{0(k+4)}, a_{1(k+4)}, b_{0(k+4)}, b_{1(k+4)}\}\right| \\ &= \left|S \cap \{a_{0(k+8)}, a_{1(k+8)}, b_{0(k+8)}, b_{1(k+8)}\}\right| \\ &= 1 \end{aligned}$$

and $|S \cap \{a_{0j}, a_{1j}, b_{0j}, b_{1j}\}| = 0$ for any $j \in \{k+1, k+2, ..., k+7\} \backslash \{k+4\}$. By Lemma 6 we may assume that $\{b_{0k}, b_{1(k+4)}, b_{0(k+8)}\} \subseteq S$. But then

$$(c(b_{0k}), c(a_{0k})) = (c(a_{1(k+4)}), c(b_{1(k+4)})) = (c(b_{0(k+8)}), c(a_{0(k+8)}))$$

and

$$(c(a_{1k}), c(b_{1k})) = (c(b_{0(k+4)}), c(a_{0(k+4)})) = (c(a_{1(k+8)}), c(b_{1(k+8)})).$$

We delete the subgraph of G induced by $\{a_{ij}, b_{ij} : 0 \leq i \leq 1, k \leq j < k+8\}$ and also delete all edges joining w to $\{b_{ij} : 0 \leq i \leq 1, k \leq j < k+8\}$. Then we join $b_{0(k-1)}$ to $a_{0(k+8)}$, $a_{0(k-1)}$ to $b_{0(k+8)}$, $a_{1(k-1)}$ to $b_{1(k+8)}$ and $b_{1(k-1)}$ to $a_{1(k+8)}$ to obtain $M(K_2 \times P_{n-8})$. Now by the hypothesis of induction $|S| - 2 \geq 2 + \left\lceil \frac{n-8-2}{4} \right\rceil$ which implies that $|S| \geq 2 + \left\lceil \frac{n-2}{4} \right\rceil$.

So $d(M(K_2 \times P_n), \chi) \geq 2 + \left\lceil \frac{n-2}{4} \right\rceil$. Now it is sufficient to define a defining set of required cardinality.

- $n \equiv 0$ $(mod\ 4)$.

  For $n \equiv 0$ $(mod\ 8)$ we define $S = \{a_{11}, b_{0(8i+3)}, b_{1(8j-1)}, b_{0n} : i \geq 0, j \geq 1\}$, where $c(a_{11}) = 2$, $c(b_{0(8k+3)}) = 2$, $c(b_{1(8k-1)}) = 3$ and $c(b_{0n}) = 3$.

  For $n \not\equiv 0$ $(mod\ 8)$ we define $S = \{a_{11}, b_{0(8i+3)}, b_{1(8j-1)}, b_{1n} : i \geq 0, j \geq 1\}$, where $c(a_{11}) = 2$, $c(b_{0(8k+3)}) = 2$, $c(b_{1(8k-1)}) = 3$ and $c(b_{1n}) = 2$.

- $n \equiv 1$ $(mod\ 4)$.

  For $n \equiv 1$ $(mod\ 8)$ we define $S = \{a_{11}, b_{0(8i+3)}, b_{1(8j-1)}, a_{0n} : i \geq 0, j \geq 1\}$, where $c(a_{11}) = 2$, $c(b_{0(8k+3)}) = 2$, $c(b_{1(8k-1)}) = 3$ and $c(a_{0n}) = 3$.

  For $n \not\equiv 1$ $(mod\ 8)$ we define $S = \{a_{11}, b_{0(8i+3)}, b_{1(8j-1)}, a_{1n} : i \geq 0, j \geq 1\}$, where $c(a_{11}) = 2$, $c(b_{0(8k+3)}) = 2$, $c(b_{1(8k-1)}) = 3$ and $c(a_{1n}) = 2$.

Then S is a defining set of the required cardinality in each case. Hence, $d(M(K_2 \times P_n), \chi) = 2 + \left\lceil \frac{n-2}{4} \right\rceil$. Hence, the the proof is complete.

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