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Two Inequalities Of Simpson Type For Quasi-Convex Functions and Applications*

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Abstract

Some inequalities of Simpson's type for quasi-convex functions in terms of third derivatives are introduced. Applications to Simpson's numerical quadrature rule is also given.

1 Introduction

Suppose $f : [a, b] \to \mathbb{R}$ is fourth times continuously differentiable function on (a, b) and

$$\left\|f^{(4)}\right\|_{\infty} := \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty.$$

Then the following inequality

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le \frac{(b-a)^5}{2880}\left\|f^{(4)}\right\|_{\infty} \tag{1}$$

holds, and in the literature known as Simpson's inequality. It is well known that if the function f is neither four times differentiable nor its fourth derivative is bounded on (a, b), then we cannot apply the classical Simpson quadrature formula.

In [13], Pečarić et al. obtained some inequalities of Simpson's type for functions whose n-th derivative, $n \in \{0, 1, 2, 3\}$ is of bounded variation, as follow:

THEOREM 1.Let $n \in \{0, 1, 2, 3\}$. Let f be a real function on $[a, b]$ such that $f^{(n)}$ is function of bounded variation. Then

$$\left|\int_a^b f(x)\,dx - \frac{(b-a)}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \le C_n (b-a)^{n+1} \bigvee_a^b \left(f^{(n)}\right), \tag{2}$$

where,

$$C_0 = \frac{1}{3},\ C_1 = \frac{1}{24},\ C_2 = \frac{1}{324},\ C_3 = \frac{1}{1152}$$

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and $\bigvee_a^b \left(f^{(n)}\right)$ is the total variation of $f^{(n)}$ on the interval $[a,b]$.

Here we note that, the inequality (2) with $n=0$, was proved by Dragomir [3]. Also, Ghizzetti et al. [9], proved that if f''' is an absolutely continuous function with total variation $\bigvee_a^b (f)$, then (2) holds with $n=3$.

In recent years many authors had established several generalizations of the Simpson's inequality for functions of bounded variation and for Lipschitzian, monotonic, and absolutely continuous functions via kernels. For refinements, counterparts, generalizations and several Simpson's type inequalities see [2]–[13] and [15]–[17].

The notion of a quasi-convex function generalizes the notion of a convex functions. More precisely, a function $f:[a,b]\to\mathbb{R}$, is said quasi-convex on $[a,b]$ if

$$f(\lambda x+(1-\lambda)y)\leq \max\{f(x),f(y)\},$$

for all $x,y\in[a,b]$ and $\lambda\in[0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are neither convex nor continuous, For more details about quasi-convex functions, we refer the reader to [14].

EXAMPLE 1. The floor function $f_{loor}(x)=\lfloor x\rfloor$, is the largest integer not greater than x, is an example of a monotonic increasing function which is quasi-convex but it is neither convex nor continuous.

In the same time, one can note that the quasi-convex functions may be not of bounded variation, i.e., there exist quasi-convex functions which are not of bounded variation. For example, consider the function $f:[0,2]\to\mathbb{R}$, defined by

$$f(x)=\begin{cases} x\sin\left(\frac{\pi}{x}\right) & \text{if } x\neq 0,\\ 0 & \text{if } x=0,\end{cases}$$

is quasi-convex but not of bounded variation on $[0,2]$. Therefore, we cannot apply the above inequalities. For new inequalities via quasi-convex function see [1, 2].

In this paper, we obtain some inequalities of Simpson type via quasi-convex function. This approach allows us to investigate Simpson's quadrature rule that has restrictions on the behavior of the integrand and thus to deal with larger classes of functions.

2 Inequalities of Simpson's Type for Quasi-Convex Functions

Let us begin with the following lemma:

LEMMA 1. Let $f'':I\subseteq\mathbb{R}\to\mathbb{R}$ be an absolutely continuous function on I° such that $f'''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'''|$ is quasi-convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned}&\int_a^b f(x)\,dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\\ =\;& (b-a)^4\int_0^1 p(t)\,f'''(ta+(1-t)b)\,dt, \qquad (3)\end{aligned}$$

where,

$$p(t)=\begin{cases}\frac{1}{6}t^2\left(t-\frac{1}{2}\right) & \text{if } t\in\left[0,\frac{1}{2}\right],\\ \frac{1}{6}(t-1)^2\left(t-\frac{1}{2}\right) & \text{if } t\in\left(\frac{1}{2},1\right].\end{cases}$$

PROOF. We note that

$$\begin{aligned} I=\int_0^1 p(t)\,f'''(ta+(1-t)b)\,dt &= \frac{1}{6}\int_0^{1/2} t^2\left(t-\frac{1}{2}\right)f'''(ta+(1-t)b)\,dt\\ &+\frac{1}{6}\int_{1/2}^1 (t-1)^2\left(t-\frac{1}{2}\right)f'''(ta+(1-t)b)\,dt. \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned} I &= \frac{1}{6}t^2\left(t-\frac{1}{2}\right)\frac{f''(ta+(1-t)b)}{a-b}\bigg|_0^{1/2}-\frac{1}{6}t(3t-1)\frac{f'(ta+(1-t)b)}{(a-b)^2}\bigg|_0^{1/2}\\ &+\left(t-\frac{1}{6}\right)\frac{f(ta+(1-t)b)}{(a-b)^3}\bigg|_0^{1/2}-\int_0^{1/2}\frac{f(ta+(1-t)b)}{(a-b)^3}dt\\ &+\frac{1}{6}(t-1)^2\left(t-\frac{1}{2}\right)\frac{f''(ta+(1-t)b)}{a-b}\bigg|_{1/2}^1\\ &-\frac{1}{6}(3t-2)(t-1)\frac{f'(ta+(1-t)b)}{(a-b)^2}\bigg|_{1/2}^1\\ &+\left(t-\frac{5}{6}\right)\frac{f(ta+(1-t)b)}{(a-b)^3}\bigg|_{1/2}^1-\int_{1/2}^1\frac{f(ta+(1-t)b)}{(a-b)^3}dt\\ &= -\frac{1}{24}\frac{f'\left(\frac{a+b}{2}\right)}{(a-b)^2}+\frac{2}{6}\frac{f\left(\frac{a+b}{2}\right)}{(a-b)^3}+\frac{1}{6}\frac{f(b)}{(a-b)^3}-\int_0^{1/2}\frac{f(ta+(1-t)b)}{(a-b)^3}dt\\ &+\frac{1}{24}\frac{f'\left(\frac{a+b}{2}\right)}{(a-b)^2}+\frac{1}{6}\frac{f(a)}{(a-b)^3}+\frac{2}{6}\frac{f\left(\frac{a+b}{2}\right)}{(a-b)^3}-\int_{1/2}^1\frac{f(ta+(1-t)b)}{(a-b)^3}dt \end{aligned}$$

Setting $x=ta+(1-t)b$, and $dx=(a-b)dt$, gives

$$(b-a)^4\cdot I=\int_a^b f(x)\,dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right],$$

which gives the desired representation (3).Therefore, we can state the following result.

THEOREM 2. Let $f'':I\subseteq\mathbb{R}\to\mathbb{R}$ be an absolutely continuous function on I° such that $f'''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'''|$ is quasi-convex on $[a,b]$, then the following inequality holds:

$$\begin{aligned} &\left|\int_a^b f(x)\,dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\ &\le \frac{(b-a)^4}{1152}\left[\max\left\{|f'''(a)|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|,|f'''(b)|\right\}\right] \quad (4) \end{aligned}$$

PROOF. From Lemma 2 and quasi-convexity of $|f'''|$, we have

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
\leq\ & (b-a)^4\int_0^1 |p(t)\,f'''(ta+(1-t)\,b)|\,dt\\
=\ & \frac{(b-a)^4}{6}\int_0^{1/2}\left|t^2\left(t-\frac{1}{2}\right)\right||f'''(ta+(1-t)\,b)|\,dt\\
&+\frac{(b-a)^4}{6}\int_{1/2}^1\left|(t-1)^2\left(t-\frac{1}{2}\right)\right||f'''(ta+(1-t)\,b)|\,dt\\
\leq\ & \frac{(b-a)^4}{6}\int_0^{1/2} t^2\left(\frac{1}{2}-t\right)\cdot\max\left\{|f'''(b)|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right\}dt\\
&+\frac{(b-a)^4}{6}\int_{1/2}^1 (1-t)^2\left(t-\frac{1}{2}\right)\cdot\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|,|f'''(a)|\right\}dt\\
=\ & \frac{(b-a)^4}{1152}\left[\max\left\{|f'''(a)|,\left|f'''\left(\frac{a+b}{2}\right)\right|\right\}+\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|,|f'''(b)|\right\}\right],
\end{aligned}$$

which completes the proof.

The corresponding version of the inequality (2.2) for powers in terms of the third derivative is incorporated as follows:

THEOREM 3. Let $f'': I\subseteq\mathbb{R}\to\mathbb{R}$ be an absolutely continuous function on I° such that $f'''\in L[a,b]$, where $a,b\in I$ with $a<b$. If $|f'''|^q$, $q=p/(p-1)$, is quasi-convex on $[a,b]$, for some fixed $p>1$, then the following inequality holds:

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\\
\leq\ & \frac{2^{-1/p}(b-a)^4}{48}\left(\frac{\Gamma(p+1)\,\Gamma(2p+1)}{\Gamma(3p+2)}\right)^{1/p}\left[\left(\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|^q,|f'''(b)|^q\right\}\right)^{1/q}\right.\\
&\left.+\left(\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|^q,|f'''(a)|^q\right\}\right)^{1/q}\right]\\
=\ & \frac{2^{-1/p}(b-a)^4}{48}\left(B(p+1,2p+1)\right)^{1/p}\left[\left(\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|^q,|f'''(b)|^q\right\}\right)^{1/q}\right.\\
&\left.+\left(\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|^q,|f'''(a)|^q\right\}\right)^{1/q}\right].
\end{aligned}$$

PROOF. From Lemma 2 and the Hölder's inequality, we have

$$
\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
\leq\; & (b-a)^4\int_0^1 |p(t)\,f'''(ta+(1-t)\,b)|\,dt \\
=\; & \frac{(b-a)^4}{6}\int_0^{1/2}\left|t^2\left(t-\frac{1}{2}\right)\right||f'''(ta+(1-t)\,b)|\,dt \\
& +\frac{(b-a)^4}{6}\int_{1/2}^1\left|(t-1)^2\left(t-\frac{1}{2}\right)\right||f'''(ta+(1-t)\,b)|\,dt \\
\leq\; & \frac{(b-a)^4}{6}\left(\int_0^{1/2}\left[t^2\left(\frac{1}{2}-t\right)\right]^p dt\right)^{1/p}\left(\int_0^{1/2}|f'''(ta+(1-t)\,b)|^q\,dt\right)^{1/q} \\
& +\frac{(b-a)^4}{6}\left(\int_{1/2}^1\left[(t-1)^2\left(t-\frac{1}{2}\right)\right]^p dt\right)^{1/p}\left(\int_{1/2}^1|f'''(ta+(1-t)\,b)|^q\,dt\right)^{1/q}.
\end{aligned}
$$

Since f is quasi-convex by Hermite-Hadamard's inequality, we have

$$\int_0^{1/2}|f'''(ta+(1-t)\,b)|^q\,dt\leq\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|^q,|f'''(b)|^q\right\},$$

and

$$\int_{1/2}^1|f'''(ta+(1-t)\,b)|^q\,dt\leq\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|^q,|f'''(a)|^q\right\}.$$

A combination of the above numbered inequalities, we get

$$
\begin{aligned}
&\left|\int_a^b f(x)\,dx-\frac{(b-a)}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right| \\
\leq\; & \frac{2^{-1/p}(b-a)^4}{48}\left(\frac{\Gamma(p+1)\,\Gamma(2p+1)}{\Gamma(3p+2)}\right)^{1/p}\left[\left(\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|^q,|f'''(b)|^q\right\}\right)^{1/q}\right. \\
& \left.+\left(\max\left\{\left|f'''\left(\frac{a+b}{2}\right)\right|^q,|f'''(a)|^q\right\}\right)^{1/q}\right],
\end{aligned}
$$

which completes the proof.

REMARK 1. Similar inequalities involving third derivative may be stated if one assumes that $|f'''|$ is convex on $[a,b]$. The details are left to the interested readers.

3 Applications to Simpson's Formula

Let d be a division of the interval $[a,b]$, i.e., $d: a = x_0 < x_1 < ... < x_{n-1} < x_n = b$, $h_i = (x_{i+1} - x_i)/2$ and consider the Simpson's formula

$$S(f,d) = \sum_{i=0}^{n-1} \frac{f(x_i) + 4f(x_i + h_i) + f(x_{i+1})}{6} (x_{i+1} - x_i).$$

It is well known that if the function $f : [a,b] \to \mathbf{R}$, is differentiable such that $f^{(4)}(x)$ exists on (a,b) and

$$M = \sup_{x\in(a,b)} \left|f^{(4)}(x)\right| < \infty,$$

then

$$I = \int_a^b f(x)\,dx = S(f,d) + E_S(f,d), \tag{5}$$

where the approximation error $E_S(f,d)$ of the integral I by the Simpson's formula $S(f,d)$ satisfies

$$|E_S(f,d)| \le \frac{M}{2880} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^5.$$

However, if the mapping f is not fourth differentiable or the fourth derivative is not bounded on (a,b), then (5) cannot be applied. In the following we give a new estimation for the remainder term $E_S(f,d)$ in terms of the third derivative.

PROPOSITION 1. Let $f'' : I \subseteq \mathbb{R} \to \mathbb{R}$ be an absolutely continuous function on I° such that $f''' \in L[a,b]$, where $a,b \in I$ with $a < b$. If $|f'''|$ is quasi-convex on $[a,b]$, then for every division d of $[a,b]$, the following holds:

$$\begin{aligned} |E_S(f,d)| \quad &\le \quad \frac{1}{1152} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^4 \left[\max\left\{f'''(x_i), f'''\left(\frac{x_i + x_{i+1}}{2}\right)\right\}\right. \\ &\left. + \max\left\{f'''\left(\frac{x_i + x_{i+1}}{2}\right), f'''(x_{i+1})\right\}\right]. \end{aligned}$$

PROOF. Applying Theorem 2 on the subintervals $[x_i, x_{i+1}]$, $(i = 0, 1, ..., n-1)$ of the division d, we get

$$\begin{aligned} &\left|\int_{x_i}^{x_{i+1}} f(x)\,dx - \frac{(x_{i+1} - x_i)}{6}\left[f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right]\right| \\ \le\ & \frac{(x_{i+1} - x_i)^4}{1152}\left[\max\left\{|f'''(x_i)|, \left|f'''\left(\frac{x_i + x_{i+1}}{2}\right)\right|\right\}\right. \\ &\left. + \max\left\{\left|f'''\left(\frac{x_i + x_{i+1}}{2}\right)\right|, |f'''(x_{i+1})|\right\}\right] \end{aligned}$$

Summing over i from 0 to $n-1$ and taking into account that $|f'''|$ is quasi-convex, we deduce that

$$\left|\int_a^b f(x)\,dx - S(f,d)\right| \le \frac{1}{1152}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^4\left[\max\left\{|f'''(x_i)|,\left|f'''\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right\}\right.$$
$$\left.+\max\left\{\left|f'''\left(\frac{x_i+x_{i+1}}{2}\right)\right|,|f'''(x_{i+1})|\right\}\right],$$

which completes the proof.

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