Applied Mathematics E-Notes, 11(2011), 101-109 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Statistical Extension Of The Korovkin-Type Approximation Theorem*

Kamil Demirci†, Fadime Dirik‡

Received 5 April 2010

Abstract

In this paper, using the concept of statistical σ-convergence which is stronger than convergence and statistical convegence we obtain a Korovkin type approximation theorem for sequences of positive linear operators from H_ω to $C_B(I)$ where $I = [0, \infty)$ and ω is a modulus of continuity type functions. Also, we construct an example such that our new approximation result works but its classical and statistical cases do not. We also compute the rates of statistical σ-convergence of sequence of positive linear operators.

1 Introduction

For a sequence $\{L_n\}$ of positive linear operators on $C(X)$, the space of real valued continuous functions on a compact subset X of real numbers, Korovkin [13] established first the sufficient conditions for the uniform convergence of $L_n(f)$ to a function f by using the test function f_i defined by $f_i(x) = x^i$, $(i = 0, 1, 2)$. Later many researchers have investigated these conditions for various operators defined on different spaces. Using the concept of statistical convergence in the approximation theory provides us with many advantages. In particular, the matrix summability methods of Cesáro type are strong enough to correct the lack of convergence of various sequences of linear operators such as the interpolation operator of Hermite-Fejér [5], because these types of operators do not converge at points of simple discontinuity. Furthermore, in recent years, with the help of the concept of uniform statistical convergence, which is a regular (non-matrix) summability transformation, various statistical approximation results have been proved [2, 3, 7, 8, 9, 11, 12]. Also, Çakar and Gadjiev have introduced the classical case of the Korovkin-type results in on the space H_ω where ω is a modulus of continuity type functions [6]. Recently various kind of statistical convergence which is stronger than the statistical convergence has been introduced by Mursaleen and Edely [15]. We first recall these convergence methods.

Let K be a subset of $\mathbb{N}$, the set of natural numbers, then the natural density of K, denoted by $\delta(K)$, is given by

$$\delta(K) := \lim_n \frac{1}{n} |\{k \le n \ : k \in K \}|$$

*Mathematics Subject Classifications: 41A25, 41A36.

†Sinop University, Faculty of Sciences and Arts, Department of Mathematics, 57000 Sinop, Turkey
‡Sinop University, Faculty of Sciences and Arts, Department of Mathematics, 57000 Sinop, Turkey

whenever the limit exists, where $|B|$ denotes the cardinality of the set B. Then a sequence $x = \{x_k\}$ of numbers is statistically convergent to L provided that, for every $\varepsilon > 0$, $\delta\{k : |x_k - L| \geq \varepsilon\} = 0$ holds ([10, 18]). In this case we write $st\text{-}\lim_k x_k = L$.

Notice that every convergent sequence is statistically convergent to the same value, but its converse is not true.

Let σ be a one-to-one mapping from the set of $\mathbb{N}$ into itself. A continuous linear functional φ defined on the space l_∞ of all bounded sequences is called an invariant mean (or σ-mean) [16] if and only if

(*i*) $\varphi(x) \geq 0$ when the sequence $x = \{x_k\}$ has $x_k \geq 0$ for all k,

(*ii*) $\varphi(e) = 1$, where $e = (1, 1, ...)$,

(*iii*) $\varphi(x) = \varphi((x_{\sigma(n)}))$ for all $x \in l_\infty$.

Thus, σ-mean extends the limit functional on c of all convergent sequences in the sense that $\varphi(x) = \lim x$ for all $x \in c$ [14]. Consequently, $c \subset V_\sigma$ where V_σ is the set of bounded sequences all of whose σ-means are equal. It is known [17] that

$$V_\sigma = \left\{ x \in l_\infty : \lim_p t_{pm}(x) = L \text{ uniformly in } m,\ L = \sigma\text{-}\lim x \right\}$$

where

$$t_{pm}(x) := \frac{x_m + x_{\sigma(m)} + x_{\sigma^2(m)} + ... + x_{\sigma^p(m)}}{p+1}.$$

We say that a bounded sequence $x = \{x_k\}$ is σ-convergent if and only if $x \in V_\sigma$. Let

$$V_\sigma^s = \left\{ x \in l_\infty :\ st - \lim_p t_{pm}(x) = L \text{ uniformly in } m,\ L = \sigma\text{-}\lim x \right\}.$$

A sequence $x = \{x_k\}$ is said to be statistically σ-convergent to L if and only if $x \in V_\sigma^s$.In this case we write $\delta(\sigma)\text{-}\lim x_k = L$. That is,

$$\lim_n \frac{1}{n} |\{p \leq n\ :\ |t_{pm}(x) - L| \geq \varepsilon\}| = 0,$$

uniformly in m. Using the above definitions, the next result follows immediately.

LEMMA 1. Statistical convergence implies statistical σ-convergence.

However, one can construct an example which guarantees that the converse of Lemma1 is not always true. Such an example was given in [15] as follows:

EXAMPLE 1.Consider the case $\sigma(n) = n + 1$ and the sequence $u = \{u_m\}$ defined as

$$u_m = \begin{cases} 1 & \text{if } m \text{ is odd,} \\ -1 & \text{if } m \text{ is even,} \end{cases} \tag{1}$$

is statistically σ-convergence ($\delta(\sigma)\text{-}\lim u_m = 0$) but it is neither convergent nor statistically convergent.

With the above terminology, the purpose of the present paper is to obtain a Korovkin-type approximation theorem for sequences of positive linear operators from H_ω to $C_B(I)$ where $I = [0, \infty)$ by means of the concept of statistical σ-convergence. Also, by considering Lemma 1 and the above Example 1, we will construct a sequence of positive linear operators such that while our new results work, their classical and statistical cases do not work. Finally, we compute the rate of statistical σ-convergence.

2 Statistical σ-Convergence of Positive Linear Operators

Throughout this paper $I := [0, \infty)$. $C(I)$ is the space of all real-valued continuous functions on I and $C_B(I) := \{f \in C(I) : f \text{ is bounded on } I\}$. The supremum norm on $C_B(I)$ is given by

$$\|f\|_{C_B(I)} := \sup_{x \in I} |f(x)|, \ (f \in C_B(I)).$$

Also, let H_ω is the space of all real valued functions f defined on I and satisfying

$$|f(x) - f(y)| \leq \omega\left(f; \left|\frac{x}{1+x} - \frac{y}{1+y}\right|\right) \tag{2}$$

where $\omega(f; \delta_1) := \omega(\delta_1)$ satisfies the following conditions (see for details [6]):

a) ω is a non-negative increasing function on $[0, \infty)$,

b) $\omega(\delta_1 + \delta_2) \leq \omega(\delta_1) + \omega(\delta_2)$

c) $\lim_{\delta_1 \to 0} \omega(\delta_1) = 0$

Let L be a linear operator from H_ω into $C_B(I)$. Then, as usual, we say that L is positive provided that $f \geq 0$ implies $L(f) \geq 0$. Also, we denote the value of $L(f)$ at a point $x \in I$ by $L(f(u); x)$ or, briefly, $L(f; x)$.

Throughout the paper, we also use the following test functions

$$f_0(u) = 1, \ f_1(u) = \frac{u}{1+u} \text{ and } f_2(u) = \left(\frac{u}{1+u}\right)^2.$$

We now recall the classical case of the Korovkin-type results introduced in [6] on the space H_ω. However, the proof also works for statistical convergence.

THEOREM 1. Let $\{L_m\}$ be a sequence of positive linear operators from H_ω into $C_B(I)$. Then, for any $f \in H_\omega$,

$$\lim \|L_m(f) - f\|_{C_B(I)} = 0$$

is satisfied if the following holds:

$$\lim \|L_m(f_i) - f_i\|_{C_B(I)} = 0, \ i = 0, 1, 2.$$

THEOREM 2. Let $\{L_m\}$ be a sequence of positive linear operators from H_ω into $C_B(I)$. Then, for any $f \in H_\omega$,

$$st\text{-}\lim \|L_m(f) - f\|_{C_B(I)} = 0$$

is satisfied if the following holds:

$$st - \lim \|L_m(f_i) - f_i\|_{C_B(I)} = 0, \ i = 0, 1, 2.$$

Now we have the following result.

THEOREM 3. Let $\{L_m\}$ be a sequence of positive linear operators from H_ω into $C_B(I)$. Then, for any $f \in H_\omega$,

$$\delta(\sigma) - \lim \|L_m(f) - f\|_{C_B(I)} = 0 \tag{3}$$

is satisfied if the following holds:

$$\delta(\sigma) - \lim \|L_m(f_i) - f_i\|_{C_B(I)} = 0,\ i = 0, 1, 2. \tag{4}$$

PROOF. Suppose that (3) holds and let $f \in H_\omega$. From (4), for every $\varepsilon > 0$, there exist $\delta_1 > 0$ such that $|f(y) - f(x)| < \varepsilon$ holds for all $y \in I$ satisfying $\left|\frac{y}{1+y} - \frac{x}{1+x}\right| < \delta_1$. Let $I_{\delta_1} := \left\{x \in I : \left|\frac{y}{1+y} - \frac{x}{1+x}\right| < \delta_1\right\}$. So we can write

$$\begin{aligned} |f(y) - f(x)| &= |f(y) - f(x)|\,\chi_{I_{\delta_1}}(y) + |f(y) - f(x)|\,\chi_{I\setminus I_{\delta_1}}(y) \\ &< \varepsilon + 2N_f\chi_{I\setminus I_{\delta_1}}(y), \end{aligned} \tag{5}$$

where χ_D denotes the characteristic function of the set D and $N_f := \|f\|_{C_B(I)}$. Also we get that

$$\chi_{I\setminus I_{\delta_1}}(y) \le \frac{1}{\delta_1^2}\left(\frac{y}{1+y} - \frac{x}{1+x}\right)^2. \tag{6}$$

Combining (5) with (6) we have

$$|f(y) - f(x)| \le \varepsilon + \frac{2N_f}{\delta_1^2}\left(\frac{y}{1+y} - \frac{x}{1+x}\right)^2. \tag{7}$$

Using linearity and positivity of the operators L_m we get, for any $m \in \mathbb{N}$, from (7), that

$$\begin{aligned} & |t_{pm}(L(f;x)) - f(x)| \\ \le\ & \frac{1}{p+1}\big(L_m(|f(u) - f(x)|;x) + L_{\sigma(m)}(|f(u) - f(x)|;x) \\ & + \cdots + L_{\sigma^p(m)}(|f(u) - f(x)|;x)\big) + |f(x)|\,|t_{pm}(L(f_0;x)) - f_0(x)| \\ \le\ & \frac{2N_f}{\delta_1^2(p+1)}\left(L_m\left(\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2;x\right) + L_{\sigma(m)}\left(\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2;x\right)\right. \\ & \left. + \cdots + L_{\sigma^p(m)}\left(\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2;x\right)\right) \\ & + \varepsilon t_{pm}(L(f_0;x)) + N_f\,|t_{pm}(L(f_0;x)) - f_0(x)| \\ \le\ & \varepsilon + \left(\varepsilon + N_f + \frac{6N_f}{\delta_1^2}\right)|t_{pm}(L(f_0;x)) - f_0(x)| + \frac{4N_f}{\delta_1^2}\,|t_{pm}(L(f_1;x)) - f_1(x)| \\ & + \frac{2N_f}{\delta_1^2}\,|t_{pm}(L(f_2;x)) - f_2(x)|. \end{aligned}$$

Then, we can write

$$\begin{aligned}\|t_{pm}(L(f))-f\|_{C_B(I)} &\le \varepsilon+K\{\|t_{pm}(L(f_0))-f_0\|_{C_B(I)}+\|t_{pm}(L(f_1))-f_1\|_{C_B(I)} \\ &\quad +\|t_{pm}(L(f_2))-f_2\|_{C_B(I)}\} \qquad (8)\end{aligned}$$

where $K:=\max\left\{\varepsilon+N_f+\frac{6N_f}{\delta_1^2},\frac{4N_f}{\delta_1^2},\frac{2N_f}{\delta_1^2}\right\}$. For a given $r>0$, choose $\varepsilon>0$ such that $\varepsilon<r$. Then, from (8),

$$\left|\left\{p\le n\middle|\,\|t_{pm}(L(f))-f\|_{C_B(I)}\ge r\right\}\right|\le\sum_{i=0}^{i}\left|\left\{p\le n\middle|\,\|t_{pm}(L(f_i))-f_i\|_{C_B(I)}\ge\frac{r-\varepsilon}{3K}\right\}\right|.$$

Therefore, using (3), we obtain (4). The proof is complete.

REMARK 1. Now we give an example such that Theorem3 works but the case of classical and statistical do not work. Suppose that $I=[0,\infty)$. We consider the following positive linear operators defined on H_ω introduced by Bleimann, Butzer and Hahn [4]:

$$T_m(f;x)=\frac{1+u_m}{(1+x)^m}\sum_{k=0}^{m}f\left(\frac{k}{m-k+1}\right)\binom{m}{k}x^k, \qquad (9)$$

where $f\in H_\omega$, $x\in I$, $m\in\mathbb{N}$ and u_m is given by (1). Now, consider the case $\sigma(n)=n+1$. If we use definition of T_m and the fact that

$$\binom{m}{k+1}=\frac{m}{k+1}\binom{m-1}{k},\quad \binom{m}{k+2}=\frac{m(m-1)}{(k+1)(k+2)}\binom{m-2}{k},$$

we can see that

$$\begin{aligned}T_m(f_0;x) &= 1+u_m\\ T_m(f_1;x) &= (1+u_m)\frac{m}{m+1}\frac{x}{1+x}\\ T_m(f_2;x) &= (1+u_m)\left(\frac{x^2}{(1+x)^2}\frac{m(m-1)}{(m+1)^2}+\frac{x}{1+x}\frac{m}{(m+1)^2}\right)\end{aligned}$$

and $\delta(\sigma)$-lim $\|T_m(f_i)-f_i\|_{C_B(I)}=0$, $i=0,1,2$. Then, observe that the sequence of positive linear operators $\{T_m\}$ defined by (9) satisfy all hypotheses of Theorem3. So, by Theorem3, we have

$$\delta(\sigma)-\lim\|T_m(f)-f\|_{C_B(I)}=0.$$

However, since $\{u_m\}$ is not convergent and statistical convergent, we conclude that classical (Theorem1) and statistical (Theorem2) versions of our result do not work for the operators T_m in (9) while our Theorem3 still works.

3 Rate of Statistical σ-Convergence

In this section, we compute the corresponding rate in statistical σ-convergence in Theorem3.

DEFINITION 1. A sequence $x = \{x_m\}$ is statistically σ-convergent to a number L with the rate of $\beta \in (0,1)$ if for every $\varepsilon > 0$,

$$\lim_n \frac{|\{p \le n : |t_{pm}(x) - L| \ge \varepsilon\}|}{n^{1-\beta}} = 0, \text{ uniformly in } m.$$

In this case, it is denoted by

$$x_m - L = o(n^{-\beta})\,(\delta(\sigma)).$$

Using this definition, we obtain the following auxiliary result.

LEMMA 2. Let $x = \{x_m\}$ and $y = \{y_m\}$ be sequences. Assume that $x_m - L_1 = o(n^{-\beta_1})\,(\delta(\sigma))$ and $y_m - L_2 = o(n^{-\beta_2})\,(\delta(\sigma))$. Then we have

(i) $(x_m - L_1) \mp (y_m - L_2) = o(n^{-\beta})\,(\delta(\sigma))$,where $\beta := \min\{\beta_1, \beta_2\}$,

(ii) $\lambda(x_m - L_1) = o(n^{-\beta_1})\,(\delta(\sigma))$, for real number λ.

PROOF. (i) Assume that $x_m - L_1 = o(n^{-\beta_1})\,(\delta(\sigma))$ and $y_m - L_2 = o(n^{-\beta_2})\,(\delta(\sigma))$. Then, for $\varepsilon > 0$, observe that

$$\begin{aligned} & \frac{|\{p \le n : \ |(t_{pm}(x) - L_1) \mp (t_{pm}(y) - L_2)| \ge \varepsilon\}|}{n^{1-\beta}} \\ \le\ & \frac{\left|\left\{p \le n : \ |t_{pm}(x) - L_1| \ge \frac{\varepsilon}{2}\right\}\right| + \left|\left\{p \le n : \ |t_{pm}(y) - L_2| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta}} \\ \le\ & \frac{\left|\left\{p \le n : \ |t_{pm}(x) - L_1| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_1}} + \frac{\left|\left\{p \le n : \ |t_{pm}(y) - L_2| \ge \frac{\varepsilon}{2}\right\}\right|}{n^{1-\beta_2}}. \end{aligned} \quad (10)$$

Now by taking the limit as $n \to \infty$ in (10) and using the hypotheses, we conclude that

$$\lim_n \frac{|\{p \le n : \ |(t_{pm}(x) - L_1) \mp (t_{pm}(y) - L_2)| \ge \varepsilon\}|}{n^{1-\beta}} = 0, \text{ uniformly in } m,$$

which completes the proof of (i). Since the proof of (ii) is similar, we omit it.

Modulus [1] is defined as follows:

$$\widetilde{\omega}(f;\delta_1) = \sup\left\{|f(u) - f(x)| : u, x \in K, \left|\frac{u}{1+u} - \frac{x}{1+x}\right| \le \delta_1\right\}, \ (f \in H_\omega).$$

It is clear that, similar to the classical modulus of continuity, $\widetilde{\omega}(f;\delta_1)$ satisfies the following conditions for all $f \in H_\omega$:

(1) $\widetilde{\omega}(f;\delta_1) \to 0$ if $\delta_1 \to 0$,

(2) $|f(u) - f(x)| \le \widetilde{\omega}(f;\delta_1)\left(1 + \frac{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2}{\delta_1^2}\right)$.

Now we have the following result.

THEOREM 4. Let $\{L_m\}$ be a sequence of positive linear operators from H_ω into $C_B(I)$. Assume that the following conditions hold:

(i) $\|L_m(f_0) - f_0\|_{C_B(I)} = o(n^{-\beta_1})(\delta(\sigma))$,

(ii) $\widetilde{\omega}(f;\alpha_{pm}) = o(n^{-\beta_2})(\delta(\sigma))$, where $\alpha_{pm} := \sqrt{\|t_{pm}(L(\varphi))\|_{C_B(I)}}$ with $\varphi(u) = \left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2$. Then, for any $f \in H_\omega$,

$$\|L_m(f) - f\|_{C_B(I)} = o(n^{-\beta})(\delta(\sigma)),$$

where $\beta := \min\{\beta_1, \beta_2\}$.

PROOF. Let $f \in H_\omega$ and $x \in I$ be fixed. Using linearity and positivity of the L_m, we have, for any $m \in \mathbb{N}$,

$$\begin{aligned}
&|t_{pm}(L(f;x)) - f(x)| \\
\leq\ & \frac{1}{p+1}\big(L_m(|f(u) - f(x)|;x) + L_{\sigma(m)}(|f(u) - f(x)|;x) \\
&+ \cdots + L_{\sigma^p(m)}(|f(u) - f(x)|;x)\big) + |f(x)|\,|t_{pm}(L(f_0;x)) - f_0(x)| \\
\leq\ & \frac{\widetilde{\omega}(f;\delta_1)}{p+1}\left(L_m\left(1 + \frac{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2}{\delta_1^2};x\right) + L_{\sigma(m)}\left(1 + \frac{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2}{\delta_1^2};x\right)\right. \\
&\left. + \cdots + L_{\sigma^p(m)}\left(1 + \frac{\left(\frac{u}{1+u} - \frac{x}{1+x}\right)^2}{\delta_1^2};x\right)\right) + N\,|t_{pm}(L(f_0;x)) - f_0(x)| \\
\leq\ & \widetilde{\omega}(f;\delta_1)\,|t_{pm}(L(f_0;x)) - f_0(x)| + \frac{\widetilde{\omega}(f;\delta_1)}{\delta_1^2} t_{pm}(L(\varphi;x)) + \widetilde{\omega}(f;\delta_1) \\
&+ N\,|t_{pm}(L(f_0;x)) - f_0(x)|,
\end{aligned}$$

where $N := \|f\|_{C_B(I)}$. Taking supremum over $x \in I$ on the both-sides of the above inequality, we obtain, for any $\delta_1 > 0$,

$$\begin{aligned}
&\|t_{pm}(L(f)) - f\|_{C_B(I)} \\
\leq\ & \widetilde{\omega}(f;\delta_1)\,\|t_{pm}(L(f_0)) - f_0\|_{C_B(I)} + \frac{\widetilde{\omega}(f;\delta_1)}{\delta_1^2}\,\|t_{pm}(L(\varphi))\|_{C_B(I)} \\
&+ \widetilde{\omega}(f;\delta_1) + N\,\|t_{pm}(L(f_0)) - f_0\|_{C_B(I)}.
\end{aligned}$$

Now if we take $\delta_1 := \alpha_{pm} := \sqrt{\|t_{pm}(L(\varphi))\|_{C_B(I)}}$, then we may write

$$\begin{aligned}
\|t_{pm}(L(f)) - f\|_{C_B(I)} \leq\ & D\Big\{\widetilde{\omega}(f;\delta_1)\,\|t_{pm}(L(f_0)) - f_0\|_{C_B(I)} + \widetilde{\omega}(f;\delta_1) \\
& + \|t_{pm}(L(f_0)) - f_0\|_{C_B(I)}\Big\} \qquad (11)
\end{aligned}$$

where $D=\max\{2,N\}$. For a given $r>0$, from (11), we get

$$\begin{aligned}
&\frac{\left|\left\{p\leq n:\|t_{pm}\left(L\left(f\right)\right)-f\|_{C_B(I)}\geq r\right\}\right|}{n^{1-\beta}}\\
\leq\quad&\frac{\left|\left\{p\leq n:\widetilde{\omega}\left(f;\delta_1\right)\geq\sqrt{\frac{r}{3D}}\right\}\right|}{n^{1-\beta_2}}+\frac{\left|\left\{p\leq n:\|t_{pm}\left(L\left(f_0\right)\right)-f_0\|_{C_B(I)}\geq\sqrt{\frac{r}{3D}}\right\}\right|}{n^{1-\beta_1}}\\
&+\frac{\left|\left\{p\leq n:\widetilde{\omega}\left(f;\delta_1\right)\geq\frac{r}{3D}\right\}\right|}{n^{1-\beta_2}}+\frac{\left|\left\{p\leq n:\|t_{pm}\left(L\left(f_0\right)\right)-f_0\|_{C_B(I)}\geq\frac{r}{3D}\right\}\right|}{n^{1-\beta_1}}.
\end{aligned}$$

Now, using (i) and (ii), we obtain

$$\lim_n\frac{\left|\left\{p\leq n:\|t_{pm}\left(L\left(f\right)\right)-f\|_{C_B(I)}\geq r\right\}\right|}{n^{1-\beta}}=0,\text{ uniformly in }m,$$

which means

$$\|L_m\left(f\right)-f\ \|_{C_B(I)}=o(n^{-\beta})\left(\delta\left(\sigma\right)\right).$$

The proof is completed.

References

[1] A. Altın, O. Doğru and M.A. Özarslan, Korovkin type approximation properties of bivariate Blemann, Butzer and Hahn operators, Proceedings of the 8th WSEAS International Conference on APPLIED MATHEMATIS, Tenerife, Span, December 16-18(2005), 234–238.

[2] G. A. Anastassiou and O. Duman, A Baskakov type generalization of statistical Korovkin theory, J. Math. Anal. Appl., 340(2008), 476–486.

[3] G. A. Anastassiou and O. Duman, Statistical fuzzy approximation by fuzzy positive linear operators, Comput. Math. Appl., 55(2008), 573–580.

[4] G. Bleimann, P. L. Butzer and L. Hahn, A Bernstein type operator approximating continuous functions on semiaxis, Indag. Math., 42(1980), 255–262.

[5] R. Bojanic and M. K. Khan, Summability of Hermite-Fejér interpolation for functions of bounded variation, J. Natur. Sci. Math. 32(1992), 5–10.

[6] Ö. Çakar and A. D. Gadjiev, On uniform approximation by Bleimann, Butzer and Hahn on all positive semiaxis, Tras. Acad. Sci. Azerb. Ser. Phys. Tech. Math. Sci. 19(1999), 21–26.

[7] O. Duman, M. K. Khan and C. Orhan, A-statistical convergence of approximating operators, Math. Inequal. Appl., 4(2003), 689–699.

[8] O. Duman, E. Erkuş and V. Gupta, Statistical rates on the multivariate approximation theory, Math. Comput. Modelling, 44(2006), 763–770.

[9] E. Erkuş, O. Duman and H. M. Srivastava, Statistical approximation of certain positive linear operators constructed by means of the Chan-Chyan-Srivastava polynomials, Appl. Math. Comput., 182(2006), 213–222.

[10] H. Fast, Sur la convergence statistique, Colloq. Math., 2(1951), 241–244.

[11] A.D. Gadjiev and C. Orhan, Some approximation theorems via statistical convergence, Rocky Mountain J. Math., 32(2002), 129–138

[12] S. Karakuş, K. Demirci and O. Duman, Equi-statistical convergence of positive linear operators, J. Math. Anal. Appl., 339(2008), 1065–1072.

[13] P.P. Korovkin, Linear Operators and Approximation Theory, Hindustan Publ. Co., Delhi, 1960.

[14] M. Mursaleen, On some new invariant matrix methods of summability, Quart. J. Math. Oxford Ser. (2) 34(1983), 77–86.

[15] M. Mursaleen and O. H. H. Edely, On invariant mean and statistical convergence, Appl. Math. Lett. 22(2009), 1700–1704.

[16] R.A. Raimi, Invariant means and invariant matrix methods of summability, Duke Math. J., 30(1963), 81–94.

[17] P. Schaefer, Infinite matrices and invariant means, Proc. Amer. Math. Soc. 36(1972), 104–110.

[18] H. Steinhaus, Sur la convergence ordinaire et la convergence asymtotique, Colloq. Math. 2(1951), 73–74.