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Value-Sharing Of Meromorphic Functions And Their Derivatives*

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Abstract

In this paper, we study the uniqueness problems on meromorphic functions concerning differential polynomials, and obtain two theorems which generalize and improve some known results.

1 Introduction

In this paper, a meromorphic function means meromorphic in the open complex plane. We shall adopt the standard notations in Nevanlinna's value distribution theory of meromorphic functions (see [1, 2]).

Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $a \in \mathbb{C}\bigcup\{\infty\}$. We say that f and g share the value a IM if $f-a$ and $g-a$ have the same zeros. Moreover, if $f-a$ and $g-a$ have the same zeros with the same multiplicities, we say that they share the value a CM. Let z_0 be the zero of $f-1$ with multiplicity p and the zero of $g-1$ with multiplicity q. We denote by $N_E^{1)}(r,1/(f-1))$ the counting function of the zeros of $f-1$ where $p=q=1$, and by $\overline{N}_L(r,1/(f-1))$ the counting function of the zeros of $f-1$ where $p>q\geq 1$; each point in these counting functions is counted only once. In the same way, we can define $N_E^{1)}(r,1/(g-1))$ and $\overline{N}_L(r,1/(g-1))$. We use $N_{p)}(r,1/(f-a))$ to denote the counting function of the zeros of $f-a$ whose multiplicities are not greater than p, and $N_{(p}(r,1/(f-a))$ to denote the counting function of the zeros of $f-a$ whose multiplicities are not less than p. Respectively, $\overline{N}_{p)}(r,1/(f-a))$ and $\overline{N}_{(p}(r,1/(f-a))$ are their reduced functions. Set

$$N_p\left(r,\frac{1}{f-a}\right)=\overline{N}\left(r,\frac{1}{f-a}\right)+\overline{N}_{(2}\left(r,\frac{1}{f-a}\right)+\cdots+\overline{N}_{(p}\left(r,\frac{1}{f-a}\right).$$

Further, we define

$$\delta_p(a,f)=1-\overline{\lim_{r\to\infty}}\frac{N_p(r,1/(f-a))}{T(r,f)}.$$

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For the sake of simplicity, we also use the notations $C_j^k = \binom{k}{j}$, and $m^* := \chi_\mu m$, where $\chi_\mu = \begin{cases} 0, & \mu = 0, \\ 1, & \mu \neq 0. \end{cases}$

Fang [4] proved the following results.

THEOREM A. Let f, g be nonconstant entire functions, and n, k be positive integers with $n > 2k + 4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f = c_1 e^{cz}$, $g = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k (c_1 c_2)^n (nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.

THEOREM B. Let f, g be nonconstant entire functions, and n, k be positive integers with $n > 2k + 8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f \equiv g$.

Recently, the authors in [5] and [6] extended Theorem A and Theorem B to meromorphic functions. In this paper, we generalize and improve the theorems above and obtain the following two theorems.

THEOREM 1. Let f, g be transcendental meromorphic functions, and n, k, m be positive integers with $n > 9k + 6m^* + 13$. If $[f^n(\mu f^m + \lambda)]^{(k)}$, $[g^n(\mu g^m + \lambda)]^{(k)}$ share 1 IM, where λ, μ are constants such that $|\lambda| + |\mu| \neq 0$, and f, g share ∞ IM,
(1) if $\lambda\mu \neq 0$, $m > 1$ and $(n, n+m) = 1$, or while $m = 1$ and $\Theta(\infty, f) > 2/n$, then $f \equiv g$;
(2) if $\lambda\mu = 0$, then either $f = tg$, where t is a constant satisfying $t^{n+m^*} = 1$ or $f = c_1 e^{cz}$, $g = c_2 e^{-cz}$, where c_1, c_2 and c are constants such that

$$(-1)^k \lambda^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1 \quad \textit{or} \quad (-1)^k \mu^2 (c_1 c_2)^{n+m^*} [(n+m^*)c]^{2k} = 1.$$

We add an example here to point out the condition $\Theta(\infty, f) > 2/n$ cannot be deleted.

EXAMPLE 1. Let $\mu = m = k = 1, \lambda = -1$, and

$$f = \frac{(n+1)(h^n - 1)h}{n(h^{n+1} - 1)}, \quad g = \frac{(n+1)(h^n - 1)}{n(h^{n+1} - 1)},$$

where $h = e^z$. Obviously, $[f^n(f-1)]'$, $[g^n(g-1)]'$ share 1 IM, and f, g share ∞ IM, $\Theta(\infty, f) = 0$, $f \not\equiv g$.

EXAMPLE 2. Let $\lambda = k = 1, \mu = m^* = 0$, and we can obtain two representations of f and $g : f = tg$ for a constant such that $t^n = 1$; $f = c_1 e^{cz}$, $g = c_2 e^{-cz}$, where c_1, c_2 and c are constants satisfying $(c_1 c_2)^n (nc)^2 = -1$.

THEOREM 2. Let f, g be transcendental meromorphic functions, and n, k, m be positive integers $n > 9k + 4m + 15$. If $[f^n(f-1)^m]^{(k)}$, $[g^n(g-1)^m]^{(k)}$ share 1 IM and f, g share ∞ IM, then either $f \equiv g$ or $f^n(f-1)^m \equiv g^n(g-1)^m$.

EXAMPLE 3. Let $m = k = 1$, and

$$f = \frac{(h^n - 1)h}{h^{n+1} - 1}, \quad g = \frac{h^n - 1}{h^{n+1} - 1},$$

where $h = e^z$. Obviously, $[f^n(f-1)]'$, $[g^n(g-1)]'$ share 1 IM, and f, g share ∞ IM, $f^n(f-1) = g^n(g-1)$.

2 Some Lemmas

In order to prove our results, we need the following lemmas.

LEMMA 1 (See [2],[7]). Let f be a nonconstant meromorphic function and n be a positive integer. then $T(r, a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f) = nT(r, f) + S(r, f)$, where a_i are meromorphic functions such that $a_n \not\equiv 0$, $T(r, a_i) = S(r, f)\,(i = 1, 2, ..., n)$.

LEMMA 2 (See [1]). Let f be a nonconstant meromorphic function and k be a positive integer, and c be a nonzero finite complex number, then

$$T(r, f) \leq \overline{N}(r, f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r, f),$$

where $N_0(r, 1/f^{(k+1)})$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

LEMMA 3 (See [1]). Let f be a transcendental meromorphic function and $\alpha_1(z)$, $\alpha_2(z)$ be meromorphic functions such that $T(r, \alpha_i) = S(r, f)\,(i = 1, 2)$, then

$$T(r, f) \leq \overline{N}(r, f) + \overline{N}\left(\frac{1}{f - \alpha_1}\right) + \overline{N}\left(\frac{1}{f - \alpha_2}\right) + S(r, f).$$

LEMMA 4 (See [8]). Let f be a nonconstant entire function and $k \geq 2$ be a positive integer. If $f \cdot f^{(k)} \neq 0$, then $f = e^{az+b}$, where $a(\neq 0)$ and b are constants.

LEMMA 5 (See [9,10]). Let f be a nonconstant meromorphic function and k be a positive integer, then

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f) \leq (p+k)\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f).$$

LEMMA 6. Let $f,\ g$ be transcendental meromorphic functions, and k be a positive integer. If $f^{(k)}$, $g^{(k)}$ share 1 IM, f, g share ∞ IM, and

$$\begin{aligned} \Delta = & (2k+3)\Theta(\infty, f) + (2k+3)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) \\ & + 2\delta_{k+1}(0, f) + 3\delta_{k+1}(0, g) > 4k + 12, \end{aligned} \tag{1}$$

then either $f^{(k)} g^{(k)} \equiv 1$ or $f \equiv g$.

Lemma 6 plays an important role in this paper, we add an example to show that the condition (1) cannot be deleted.

EXAMPLE 4. Let $f = -\frac{1}{2}e^{2z} - \frac{1}{2}e^{z}$, $\quad g = \frac{1}{2}e^{-2z} + \frac{1}{2}e^{-z}$. Obviously, f' g' share 1 IM, and f, g share ∞ IM. Since $T(r, f) = 2T(r, e^z) + S(r, e^z)$, and $N(r, \frac{1}{f}) = N(r, \frac{1}{e^z+1})$. The second main theorem gives $T(r, e^z) \leq \overline{N}(r, \frac{1}{e^z}) + \overline{N}(r, \frac{1}{e^z+1}) + S(r, e^z)$, so $T(r, e^z) = N(r, \frac{1}{e^z+1}) + S(r, e^z)$, and $\delta(0, f) = 1/2$, but $f \not\equiv g,\ f'g' \not\equiv 1$.

PROOF of Lemma 6. Let

$$h(z) = \Big(\frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)} - 1}\Big) - \Big(\frac{g^{(k+2)}}{g^{(k+1)}} - 2\frac{g^{(k+1)}}{g^{(k)} - 1}\Big). \tag{2}$$

If $h(z) \not\equiv 0$, and suppose that z_0 is a common simple 1-point of $f^{(k)}$ and $g^{(k)}$, then by (2), we can get $h(z_0) = 0$, and

$$N_E^{1)}\left(r, \frac{1}{f^{(k)}-1}\right) = N_E^{1)}\left(r, \frac{1}{g^{(k)}-1}\right) \le \overline{N}\left(r, \frac{1}{h}\right) \le N(r,h) + S(r,f) + S(r,g). \quad (3)$$

By assumptions, we deduce from (2) that

$$\begin{aligned} N(r,h) \quad \le \quad & \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + \overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) \\ & + \overline{N}_0(r, \frac{1}{f^{(k+1)}}) + \overline{N}_0(r, \frac{1}{g^{(k+1)}}), \end{aligned} \quad (4)$$

where $N_0(r, 1/f^{(k+1)})$ has the same meaning as in Lemma 2, and we have

$$\begin{aligned} T(r,f) + T(r,g) \quad \le \quad & \overline{N}(r,f) + \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{f}\right) + N_{k+1}\left(r, \frac{1}{g}\right) \\ & + \overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) \\ & - N_0\left(r, \frac{1}{f^{(k+1)}}\right) - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,f) + S(r,g). \end{aligned} \quad (5)$$

Since $f^{(k)}$ and $g^{(k)}$ share 1 IM , we find

$$\begin{aligned} & \overline{N}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}\left(r, \frac{1}{g^{(k)}-1}\right) \\ \le \quad & N_E^{1)}\left(r, \frac{1}{f^{(k)}-1}\right) + \overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) + N\left(r, \frac{1}{f^{(k)}-1}\right) \\ \le \quad & N_E^{1)}(r, \frac{1}{f^{(k)}-1}) + \overline{N}_L(r, \frac{1}{g^{(k)}-1}) + T(r,f) + k\overline{N}(r,f) + S(r,f). \end{aligned} \quad (6)$$

By Lemma 5, we get

$$\overline{N}\left(r, \frac{1}{f^{(k)}}\right) \le N_{k+1}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f), \quad (7)$$

and

$$\overline{N}_L\left(r, \frac{1}{f^{(k)}-1}\right) \le (k+1)\overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f}\right) + S(r,f). \quad (8)$$

In the same way, we have

$$\overline{N}_L\left(r, \frac{1}{g^{(k)}-1}\right) \le (k+1)\overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + S(r,g). \quad (9)$$

From (3)-(9), we obtain

$$\begin{aligned} T(r,g) \quad \le \quad & (2k+3)\overline{N}(r,f) + (2k+3)\overline{N}(r,g) + \overline{N}\left(r, \frac{1}{f}\right) \\ & + \overline{N}\left(r, \frac{1}{g}\right) + 2N_{k+1}\left(r, \frac{1}{f}\right) + 3N_{k+1}\left(r, \frac{1}{g}\right) + S(r,f) + S(r,g). \end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite linear measure such that $T(r,f) \leq T(r,g)$ for $r \in I$, then we deduce

$$\begin{aligned} T(r,g) &\leq [(2k+3)(1-\Theta(\infty,f)) + (2k+3)(1-\Theta(\infty,g)) + (1-\Theta(0,f)) \\ &\quad +(1-\Theta(0,g)) + 2(1-\delta_{k+1}(0,f)) + 3(1-\delta_{k+1}(0,g)) + \varepsilon]T(r,g) + S(r,g) \end{aligned}$$

for $r \in I$ and $0 < \varepsilon < \Delta - (4k+12)$, that is

$$[\Delta - (4k+12) - \varepsilon]T(r,g) \leq S(r,g),$$

this together with (1) may lead to a contradiction. Hence $h(z) \equiv 0$, that is

$$\frac{f^{(k+2)}}{f^{(k+1)}} - 2\frac{f^{(k+1)}}{f^{(k)}-1} = \frac{g^{(k+2)}}{g^{(k+1)}} - 2\frac{g^{(k+1)}}{g^{(k)}-1}.$$

Integration yields

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)}+a-b}{g^{(k)}-1}, \tag{10}$$

where $a\,(a \neq 0)$ and b are constants. Next, we consider three cases.

Case 1. If $b = 0$. Then from (10), we obtain

$$f = g/a + p(z), \tag{11}$$

where $p(z)$ is a polynomial.

If $p(z) \not\equiv 0$, since f is transcendental, then by Lemma 3, we have

$$T(r,f) \leq \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + S(r,f). \tag{12}$$

It follows from (11) and (12) that

$$T(r,f) \leq \{3 - [\Theta(\infty,f) + \Theta(0,f) + \Theta(0,g)] + \varepsilon\}T(r,f) + S(r,g),$$

where $0 < \varepsilon < (2k+2)(1-\Theta(\infty,f)) + (2k+3)(1-\Theta(\infty,g)) + 2(1-\delta_{k+1}(0,f)) + 3(1-\delta_{k+1}(0,g))$. Therefore $T(r,f) \leq \{4k+13-\Delta\}T(r,f) + S(r,f)$, which and (1) lead to $T(r,f) \leq S(r,f)$ for $r \in I$, a contradiction. So $p(z) \equiv 0$, this yields $a = 1$, and $f \equiv g$.

Case 2. Suppose that $b \neq 0$ and $a \neq b$.

If $b = -1$, then from (10), we have $\overline{N}(r, 1/(g^{(k)} - a - 1)) = \overline{N}(r, f^{(k)}) = \overline{N}(r,f)$. Lemma 2 gives

$$\begin{aligned} T(r,g) &\leq \overline{N}(r,g) + N_{k+1}\left(r,\frac{1}{g}\right) + \overline{N}\left(r,\frac{1}{g^{(k)}-(a+1)}\right) - N_0\left(r,\frac{1}{g^{(k+1)}}\right) + S(r,g) \\ &\leq (2k+3)\overline{N}(r,f) + (2k+3)\overline{N}(r,g) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,\frac{1}{g}) \\ &\quad +2N_{k+1}\left(r,\frac{1}{f}\right) + 3N_{k+1}\left(r,\frac{1}{g}\right) + S(r,f) + S(r,g), \end{aligned}$$

which implies $T(r,g) \le \{4k+13-\Delta\}T(r,g)+S(r,g)$, and $T(r,g) \le S(r,g)$ for $r \in I$, a contradiction, so $b \ne -1$, it follows from (10) that

$$f^{(k)} - (1+1/b) = \frac{-a}{b^2[g^{(k)} + (a-b)/b]},$$

and

$$\overline{N}\left(r, \frac{1}{g^{(k)} + (a-b)/b}\right) = \overline{N}(r, f^{(k)} - (1+1/b)) = \overline{N}(r,f).$$

Similarly by Lemma 2, we have

$$\begin{aligned} T(r,g) &\le \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{g^{(k)} + (a-b)/b}\right) \\ &\quad - N_0\left(r, \frac{1}{g^{(k+1)}}\right) + S(r,g) \\ &\le (2k+3)\{\overline{N}(r,f) + \overline{N}(r,g)\} + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) \\ &\quad + 2N_{k+1}\left(r, \frac{1}{f}\right) + 3N_{k+1}\left(r, \frac{1}{g}\right) + S(r,g). \end{aligned}$$

Using the argument as in Case 2, we can also get a contradiction.

Case 3. Suppose that $b \ne 0$ and $a = b$.

If $b \ne -1$, from (10), we have

$$\overline{N}\left(r, \frac{1}{g^{(k)} - 1/(1+b)}\right) = \overline{N}\left(r, \frac{1}{f^{(k)}}\right).$$

From (7) we get

$$\overline{N}\left(r, \frac{1}{g^{(k)} - 1/(1+b)}\right) \le \overline{N}_{k+1}\left(r, \frac{1}{f}\right) + k\overline{N}(r,f) + S(r,f).$$

Lemma 2 means that

$$T(r,g) \le \overline{N}(r,g) + N_{k+1}\left(r, \frac{1}{g}\right) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + S(r,g).$$

Using the argument as in Case 2, a contradiction can also be obtained. Therefore $b = -1$, and (10) implies $f^{(k)}g^{(k)} \equiv 1$. Thus we get the conclusion of Lemma 6.

3 Proof of Theorem 1

Set $F(z) = f^n(\mu f^m + \lambda)$ and $G(z) = g^n(\mu g^m + \lambda)$, Lemma 1 gives

$$\Theta(0,F) = 1 - \lim_{r\to\infty} \sup \frac{\overline{N}(r, 1/f^n) + \overline{N}(r, 1/(\mu f^m + \lambda))}{(n+m^*)T(r,f)} \ge 1 - \frac{1+m^*}{n+m^*}. \qquad (13)$$

Similarly,

$$\Theta(0,G) \geq 1 - \frac{1+m^*}{n+m^*}, \tag{14}$$

$$\Theta(\infty,F) = 1 - \lim_{r\to\infty} \sup \frac{\overline{N}(r,f)}{(n+m^*)T(r,f)} \geq 1 - \frac{1}{n+m^*}. \tag{15}$$

In the same manner as above, we obtain

$$\Theta(\infty,G) \geq 1 - \frac{1}{n+m^*}. \tag{16}$$

$$\begin{aligned}\delta_{k+1}(0,F) &\geq 1 - \lim_{r\to\infty} \sup \frac{N_{k+1}(r,1/f^n) + N_{k+1}(r,1/(\mu f^m+\lambda))}{(n+m^*)T(r,f)} \\ &\geq 1 - \frac{k+1+m^*}{n+m^*}. \end{aligned} \tag{17}$$

And

$$\delta_{k+1}(0,G) \geq 1 - \frac{k+1+m^*}{n+m^*}. \tag{18}$$

From (13)-(18), we get

$$\begin{aligned}\Delta &= (2k+3)\Theta(\infty,F) + (2k+3)\Theta(\infty,G) + \Theta(0,F) \\ &\quad + \Theta(0,G) + 2\delta_{k+1}(0,F) + 3\delta_{k+1}(0,G) \\ &\geq 4k+13 - [(9k+13+7m^*)/(n+m^*)].\end{aligned}$$

Note that $n > 9k+13+6m^*$, we deduce that $\Delta > 4k+12$.

By Lemma 6, we deduce that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$. Next, we consider two cases.

Case 1. $F^{(k)}G^{(k)} \equiv 1$. That is

$$[f^n(\mu f^m+\lambda)]^{(k)}[g^n(\mu g^m+\lambda)]^{(k)} \equiv 1. \tag{19}$$

If $\lambda\mu = 0$. Lemma 4 and (19) give $f(z) = c_1e^{cz}, g(z) = c_2e^{-cz}$, where c_1, c_2 and c are constants satisfying $(-1)^k\lambda^2(c_1c_2)^{n+m^*}[(n+m^*)c]^{2k} = 1$ *or* $(-1)^k\mu^2(c_1c_2)^{n+m^*}[(n+m^*)c]^{2k} = 1$, for all positive integers k.

If $\lambda\mu \neq 0$. Since f, g share ∞ IM and (19), we see that f is an entire function and

$$[f^n(\mu f^m+\lambda)]^{(k)} \neq 0,\infty, \quad [g^n(\mu g^m+\lambda)]^{(k)} \neq 0,\infty. \tag{20}$$

Let $f = e^{\alpha(z)}$, where $\alpha(z)$ is a nonconstant entire function. By induction, we have

$$[\mu f^{n+m}(z)]^{(k)} = q_1(\alpha',\alpha'',...,\alpha^{(k)})e^{(m+n)\alpha(z)}, \quad [\lambda f^n(z)]^{(k)} = q_2(\alpha',\alpha'',...,\alpha^{(k)})e^{n\alpha(z)},$$

where $q_i(\alpha',\alpha'',...,\alpha^{(k)})$ $(i=1,2)$ are differential polynomials.

Note that (20) and $\lambda \neq 0$, $\mu \neq 0$, we find

$$q_1(\alpha',\alpha'',...,\alpha^{(k)})e^{m\alpha(z)} + q_2(\alpha',\alpha'',...,\alpha^{(k)}) \neq 0, \tag{21}$$

and

$$T(r,\alpha') = m(r,\alpha') = m\left(r, \frac{(e^{\alpha})'}{e^{\alpha}}\right) = m\left(r, \frac{f'}{f}\right) = S(r,f).$$

Thus

$$T(r,\alpha^{(j)}) \le T(r,\alpha') + S(r,f) = S(r,f) \text{ for } j = 1,2,...,k.$$

$$T(r,q_1) = S(r,f),\ T(r,q_2) = S(r,f).$$

By Lemma 1, Lemma 3 and (21), we get $T(r,f) \le T(r, q_1 e^{m\alpha(z)}) + S(r,f) = S(r,f)$, which is a contradiction.

Case 2. $F \equiv G$. That is

$$f^n(\mu f^m + \lambda) \equiv g^n(\mu g^m + \lambda). \tag{22}$$

If $\lambda\mu = 0$, it follows from $|\lambda| + |\mu| \ne 0$ and (22) that $f = tg$, where t is a constant such that $t^{n+m^*} = 1$.

If $\lambda\mu \ne 0$, let $f/g = H$ be not a constant, substituting $f = gH$ into (22), we have

$$mT(r,f) = T(r,f^m) + S(r,f) = (n+m-1)T(r,H) + S(r,f).$$

The second main theorem gives

$$\overline{N}(r,f) = \sum_{j=1}^{n+m-1} \overline{N}\left(r, \frac{1}{H - a_j}\right) \ge (n+m-3)T(r,H) + S(r,f),$$

where $(a_j \ne 1)\,(j = 1,2,\cdots,n+m-1)$ are distinct roots of $H^{n+m} = 1$, and we find

$$\begin{aligned}\Theta(\infty,f) &= 1 - \lim_{r\to\infty}\sup \frac{(n+m-3)T(r,H)}{T(r,f)}\\ &\le 1 - \frac{m(n+m-3)}{n+m-1} = (1-m) + \frac{2\,m}{n+m-1}.\end{aligned}$$

If $m = 1$, then $\Theta(\infty,f) \le 2/n$, a contradiction.

If $m > 1$, note that $n > 9k + 13 + 6m$, then $\Theta(\infty,f) < 0$, this is impossible. So H is a constant. If $H \not\equiv 1$, we deduce g is a constant, which is a contradiction, thus $f \equiv g$. This completes the proof of Theorem 1.

4 Proof of Theorem 2

Set $F(z) = f^n(f-1)^m$ and $G(z) = g^n(g-1)^m$, we obtain

$$\Theta(0,F) \ge 1 - \frac{2}{n+m}, \quad \Theta(0,G) \ge 1 - \frac{2}{n+m}. \tag{23}$$

$$\Theta(\infty,F) = 1 - \lim_{r\to\infty}\sup \frac{\overline{N}(r,f)}{(n+m)T(r,f)} \ge 1 - \frac{1}{n+m}. \tag{24}$$

Likewise,

$$\Theta(\infty,G) \ge 1 - \frac{1}{n+m}. \tag{25}$$

Next, we have

$$\delta_{k+1}(0,F) \geq 1 - \frac{k+1+m}{n+m}, \quad \delta_{k+1}(0,G) \geq 1 - \frac{k+1+m}{n+m}. \tag{26}$$

From (23)-(26), we get

$$\begin{aligned}\Delta &= (2k+3)\Theta(\infty,F) + (2k+3)\Theta(\infty,G) + \Theta(0,F) \\ &\quad + \Theta(0,G) + 2\delta_{k+1}(0,F) + 3\delta_{k+1}(0,G) \\ &\quad \geq 4k + 13 - (9k+15+5m)/(n+m).\end{aligned}$$

Note that $n > 9k + 15 + 4m$, we deduce that $\Delta > 4k + 12$. Lemma 6 shows that either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$. Next, we consider two cases.

Case 1. $F^{(k)}G^{(k)} \equiv 1$. That is

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv 1. \tag{27}$$

By the same argument as proof in Theorem 1, we see that (27) does not hold.

Case 2. $F \equiv G$. That is

$$f^n(f^m + \cdots + (-1)^i C_m^{m-i} f^{m-i} + \cdots + (-1)^m) \equiv g^n(g^m + \cdots + (-1)^i C_m^{m-i} g^{m-i} + \cdots + (-1)^m).$$

Let $H = f/g$, if H is a constant, substituting $f = gH$ into the equality above, we deduce $H \equiv 1$, and then $f \equiv g$.

If H is not a constant, it follows from $F \equiv G$ that $f^n(f-1)^m \equiv g^n(g-1)^m$. This completes the proof of Theorem 2.

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References

[1] W. K. Hayman, Meromorphic functions, Clarendon Press, Oxford, 1964.

[2] H. X. Yi and C. C. Yang, Uniqueness Theory of Meromorphic Functions, Science Press, Beijing, 1995.

[3] X. Y. Zhang and W. C. Lin, Uniqueness and value-sharing of entire functions, J. Math. Anal. Appl., 343(2008), 938–950.

[4] M. L. Fang, Uniqueness and value-sharing of entire functions, Comput. Math. Appl., 44(2002), 823–831.

[5] S. S. Bhoosnurmath and R. S. Dyavanal, Uniqueness and value-sharing of meromorphic function, Comput. Math. Appl., 53(2007), 1191–1205.

[6] T. D. Zhang and W. R. Lü, Uniqueness theorems on meromorphic functions sharing one value, Comput. Math. Appl., 55(2008), 2981–2992.

[7] C. C. Yang and X. H. Hua, Uniqueness and value-sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math., 22(2)(1997), 395–406.

[8] G. Frank, Eine Vermutung von Hayman uber nullstellen meromorpher funktion, Math. Z., 149(1976), 29–36.

[9] H. H. Chen, Yosida functions and Picard values of integral function and their derivatives, Bull. Austral. Math. Soc., 54(3)(1996), 373–381.

[10] Q. C. Zhang, Meromorphic function that shares one small function with its derivative, J. Inequal. Pure Appl. Math., 6(4)(2005), Art., 116.