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Asymptotic Behaviour Of Solutions For Some Weakly Dissipative Wave Equations Of p-Laplacian Type*

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Abstract

In this paper we study decay properties of some weakly dissipative wave equations of p-Laplacian type.

1 Introduction

We consider the initial boundary problem for the nonlinear wave equation of p-Laplacian type with a weak nonlinear dissipation of the type

$$\begin{cases} u_{tt} - \Delta_p u + \sigma(t)(u_t + \mid u_t \mid^{m-2} u_t) = 0, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) \text{ in } \Omega. \end{cases} \tag{1}$$

where $\Delta_p u = div(|\nabla_x u|^{p-2}\nabla_x u)$, $p \geq 2$, σ is a positive function and Ω is a bounded domain of $\mathbb{R}^n (n \geq 1)$, with a smooth boundary $\Gamma = \partial\Omega$.

For the problem (1), when $p = 2$ and $\sigma \equiv 1$, Messaoudi [7] showed that, for any initial data $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, the problem has a unique global solution with energy decaying exponentially. In the case when $g(u_t) = |u_t|^{m-2}u_t$, Nakao [9] showed that (1) has a unique global weak solution if $0 \leq \theta - 2 \leq 2/(n-3), n \geq 3$ and a global unique strong solution if $\theta - 2 > 2/(n-2), n \geq 3$ (of course if $n = 1$ or 2 then the only requirement is $\theta \geq 2$). In addition to global existence the issue of the decay rate was also addressed. In both cases it has been shown that the energy of the solution decays algebraically if $m > 2$ and decays exponentially if $m = 2$. This improves an earlier result by Nakao [10], where he studied the problem in an abstract setting and established a theorem concerning decay of the solution energy only for the case $m - 2 \leq 2/(n-2), n \geq 3$.

Our purpose in this paper is to give an energy decay estimates of the solutions to the problem (1) for a weak nonlinear dissipation, we extend the results obtained by Ye [16], also we prove in some cases an exponential decay when $p > 2$ and the dissipative term is not necessarily superlinear near the origin.

We use a new method recently introduced by Martinez [6] (see also [2]) to study the decay rate of solutions to the wave equation $u'' - \Delta_x u + g(u') = 0$ in $\Omega \times \mathbb{R}^+$, where

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Ω is a bounded domain of $\mathbb{R}^n$. This method is based on new integral inequality that generalizes a result of Haraux [4].

Throughout this paper the functions considered are all real valued. We omit the space variable x of $u(x,t)$, $u_t(x,t)$ and simply denote $u(x,t)$, $u_t(x,t)$ by $u(t)$, $u'(t)$, respectively, when no confusion arises. Let l be a number with $2 \le l \le \infty$. We denote by $\| \, . \, \|_l$ the L^l norm over Ω. In particular, L^2 norm is denoted $\| \, . \, \|_2$. $(\, . \,)$ denotes the usual L^2 inner product. We use familiar function spaces $W_0^{1,p}$.

2 Preliminaries and Main Results

The function $\sigma(t)$ satisfies the following hypotheses:

(H1) $\sigma : \mathbb{R}_+ \to \mathbb{R}_+$ is a nonincreasing function of class C^1 on $\mathbb{R}_+$ satisfying

$$\int_0^{+\infty} \sigma(\tau)\, d\tau = +\infty.$$

We define the energy associated to the solution of (1) by the following formula

$$E(t) = \frac{1}{2}\|u'\|_2^2 + \frac{1}{p}\|\nabla_x u\|_p^p.$$

We first state two well known lemmas, and then we state and prove a lemma that will be needed later.

LEMMA 1 (Sobolev-Poincaré inequality). Let q be a number with $2 \le q < +\infty$ $(n \le p)$ or $2 \le q \le \frac{np}{n-p}$ $(n \ge p+1)$, then there is a constant $c_* = c(\Omega, q)$ such that

$$\|u\|_q \le c_* \|\nabla u\|_p \quad \text{for} \quad u \in W_0^{1,p}(\Omega).$$

LEMMA 2 ([5]). Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there are two constants $q \ge 0$ and $A > 0$ such that

$$\int_S^{+\infty} E^{q+1}(t)dt \le \frac{1}{A} E(0)^q E(S), \quad 0 \le S < +\infty.$$

Then we have

$$E(t) \le E(0)\left(\frac{1+q}{1+qAt}\right)^{1/q} \quad \forall t \ge 0, \quad \text{if} \quad q > 0$$

and

$$E(t) \le E(0)e^{1-At} \;\; \forall t \ge 0, \quad \text{if} \quad q = 0.$$

LEMMA 3 ([6]). Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ an increasing C^2 function such that

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \to +\infty \quad \text{as} \quad t \to +\infty.$$

Assume that there exist $q \geq 0$ and $A > 0$ such that

$$\int_S^{+\infty} E(t)^{q+1}(t)\phi'(t)\,dt \leq \frac{1}{A}E(0)^q E(S), \quad 0 \leq S < +\infty.$$

Then we have

$$E(t) \leq E(0)\left(\frac{1+q}{1+qA\phi(t)}\right)^{1/q} \quad \forall t \geq 0, \quad \text{if} \quad q > 0$$

and

$$E(t) \leq cE(0)e^{-\omega\phi(t)} \;\; \forall t \geq 0, \quad \text{if} \quad q = 0.$$

PROOF. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be defined by $f(x) := E(\phi^{-1}(x))$, (we remark that ϕ^{-1} has a sense by the hypotheses assumed on ϕ). The function f is non-increasing, $f(0) = E(0)$ and if we set $x := \phi(t)$ we obtain

$$\begin{aligned}
\int_{\phi(S)}^{\phi(T)} f(x)^{q+1}dx &= \int_{\phi(S)}^{\phi(T)} E\left(\phi^{-1}(x)\right)^{q+1} dx \\
&= \int_S^T E(t)^{q+1}\phi'(t)dt \\
&\leq \frac{1}{A}E(0)^q E(S) \\
&= \frac{1}{A}E(0)^q f(\phi(S)), \quad 0 \leq S < T < +\infty.
\end{aligned}$$

Setting $s := \phi(S)$ and letting $T \to +\infty$, we deduce that

$$\int_s^{+\infty} f(x)^{q+1}\,dx \leq \frac{1}{A}E(0)^q f(s), \quad 0 \leq s < +\infty.$$

Thanks to Lemma 2, we deduce the desired results.

Now we recall the following global existence, which can be established by using the argument in [9].

THEOREM 1. Assume that $(u_0, u_1) \in \mathcal{W}_0^{1,p}(\Omega) \times L^2(\Omega)$. Then the problem (1) admits a unique strong solution on $\Omega \times [0,\infty)$ in the class

$$C([0,\infty[,\mathcal{W}_0^{1,p}(\Omega)) \cap C^1([0,\infty), L^2(\Omega)).$$

Our main result is the following.

THEOREM 2. Let $(u_0, u_1) \in W_0^{1,p} \times L^2(\Omega)$, $2 < m \leq \frac{2n}{(n-2)^+}$ and suppose that $(H1)$ holds. Then the solution $u(x,t)$ of the problem (1) satisfies the following energy decay estimates.

(1) If $p = 2$, then there exists a positive constant ω such that

$$E(t) \leq C(E(0))exp\left(1 - \omega\int_0^t \sigma(\tau)\,d\tau\right) \quad \forall t > 0.$$

(2) If $p > 2$, then there exists a positive constant $C(E(0))$ depending continuously on $E(0)$ such that

$$E(t) \leq \left(\frac{C(E(0))}{\int_0^t \sigma(\tau)\, d\tau} \right)^{\frac{p}{p-2}}, \quad \forall t > 0.$$

EXAMPLES.

1) Suppose that $\sigma(t) = \frac{1}{t^\theta}$ $(0 \leq \theta \leq 1)$, by applying Theorem 2 we obtain

$$E(t) \leq C(E(0))e^{1-\omega t^{1-\theta}} \text{ if } \theta \in [0,1),\ p = 2,$$

$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{p-2}} \text{ if } 0 \leq \theta < 1,\ p > 2$$

and

$$E(t) \leq C(E(0))(\ln t)^{-\frac{p}{p-2}} \text{ if } \theta = 1,\ l < m+1.$$

2) Suppose that $\sigma(t) = \frac{1}{t^\theta \ln t \ln_2 t \cdots \ln_k t}$, where k is a positive integer and

$$\begin{cases} \ln_1(t) = \ln(t), \\ \ln_{k+1}(t) = \ln(\ln_k(t)), \end{cases}$$

by applying Theorem 2, we obtain

$$E(t) \leq C(E(0))(\ln_{k+1} t)^{-\frac{p}{p-2}} \text{ if } \theta = 1,\ p > 2,$$

$$E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{p-2}}(\ln t \ln_2 t \cdots \ln_k t)^{\frac{p}{p-2}} \text{ if } 0 \leq \theta < 1,\ p > 2.$$

3) Suppose that $\sigma(t) = \frac{1}{t^\theta (\ln t)^\gamma}$, by applying Theorem 2, we obtain

$$\begin{cases} E(t) \leq C(E(0))t^{-\frac{(1-\theta)p}{p-2}}(\ln t)^{\frac{\gamma p}{p-2}} & \text{if } 0 \leq \theta < 1, p > 2, \\ E(t) \leq C(E(0))(\ln t)^{-\frac{(1-\gamma)p}{p-2}} & \text{if } \theta = 1,\ 0 \leq \gamma < 1, p > 2, \\ E(t) \leq C(E(0))(\ln_2 t)^{-\frac{p}{p-2}} & \text{if } \theta = 1,\ \gamma = 1, \quad p > 2. \end{cases}$$

3 Proof of Theorem 2

First we have the following energy identity for the problem (1).

LEMMA 4 (Energy identity). Let $u(t,x)$ be a global solution to the problem (1) on $[0,\infty)$ as in Theorem 1. Then we have

$$E(t) + \int_\Omega \int_0^t \sigma(s)u'(s)g(u'(s))dsdx = E(0)$$

for all $t \in [0,\infty)$ and where we set $g(u') = u' + |u'|^{m-2}\, u'$.

Now, we shall derive the decay estimate for the solutions in Theorem 1. For this we use the method of multipliers. We denote by c various positive constants which may be different at different occurrences.

We multiply the first equation of (1) by $E^q\phi' u$, where ϕ is a function satisfying all the hypotheses of Lemma 3. We obtain

$$\begin{aligned}
0 &= \int_S^T E^q\phi' \int_\Omega u\Big(u'' - \Delta_p u + \sigma(t)(u' + |u'|^{m-2}u')\Big)\,dxdt \\
&= \int_S^T E^q\phi' \int_\Omega uu'' dxdt - \int_S^T E^q\phi' \int_\Omega u\Delta_p u dxdt \\
&\quad + \int_S^T E^q\phi' \int_\Omega \sigma(t)u(u' + |u'|^{m-2}u')dxdt \\
&= \left[E^q\phi' \int_\Omega uu'dx\right]_S^T - \int_S^T (qE'E^{q-1}\phi' + E^q\phi'') \int_\Omega uu'dxdt \\
&= -\int_S^T E^q\phi' \int_\Omega |u'|^2\, dxdt + \int_S^T E^q\phi' \int_\Omega |\nabla u|^p\, dxdt \\
&\quad + \int_S^T E^q\phi' \int_\Omega \sigma(t)u(u' + |u'|^{m-2}u')dxdt.
\end{aligned}$$

We deduce that

$$\begin{aligned}
2\int_S^T E^{q+1}\phi' dt &= -\left[E^q\phi' \int uu'dx\right]_S^T + \int_S^T (qE'E^{q-1}\phi' + E^q\phi'') \int_\Omega uu'dxdt \\
&\quad +2\int_S^T E^q\phi' \int_\Omega u'^2 dxdt + \left(\frac{2}{p} - 1\right)\int_S^T E^q\phi' \int_\Omega |\nabla u|^p\, dxdt \\
&\quad + \int_S^T E^q\phi' \int_\Omega \sigma(t)u(u' + |u'|^{m-2}u')dxdt
\end{aligned} \tag{2}$$

Define

$$\phi(t) = \int_0^t \sigma(s)ds.$$

It is clear that ϕ is a non decreasing function of class C^2 on $\mathbb{R}_+$. Hypothesis (H1) ensures that

$$\phi(t) \to +\infty \text{ as } t \to +\infty. \tag{3}$$

Since E is non-increasing, ϕ' is a bounded non-negative function on $\mathbb{R}_+$ (and we denote by μ its maximum), we find that

$$\left|E(t)^q\phi' \int_\Omega uu'\,dx\right| \le \left[-cE^{q+\frac{1}{p}+\frac{1}{2}}\phi'\right]_S^T \le c\mu E(S)^{q+\frac{1}{2}+\frac{1}{p}}, \qquad \forall t \ge S,$$

$$\left|(qE'E^{q-1}\phi' + E^q\phi'') \int_\Omega uu'\,dx\right| \;\le\; c\mu \int_S^T (-E'(t))E(t)^{q-\frac{1}{2}+\frac{1}{p}}dt$$

$$+c\int_S^T E(t)^{q+\frac{1}{2}+\frac{1}{p}}(-\phi'')dt$$
$$\leq c\mu E(S)^{q+\frac{1}{2}+\frac{1}{p}},$$

$$\begin{aligned}
2\int_S^T E^q\phi'\int_\Omega u'^2dxdt &\leq 2\int_S^T E^q\frac{\phi'}{\sigma(t)}\int_\Omega \sigma(t)(u'^2+|u'|^m)dxdt \\
&\leq -\int_S^T E^q(t)E'(t)dt \\
&\leq C'E^{q+1}(S)
\end{aligned}$$

where we have also used the Hölder and Sobolev-Poincaré inequalities. Using these estimates we conclude from (2) that

$$\begin{aligned}
2\int_S^T E(t)^{1+q}\phi'(t)dt &\leq c\mu E(S)^{q+\frac{1}{2}+\frac{1}{p}} + c'E(S)^{q+1} \\
&\quad +\int_S^T E^q\phi'\int_\Omega \sigma(t)u(u'+|u'|^{m-2}u')dxdt. \qquad (4)
\end{aligned}$$

Now, we estimate the terms of the right-hand side of (4) in order to apply the results of Lemma 3:

$$\begin{aligned}
&\int_S^T E^q\phi'\int_\Omega \sigma(t)u(u'+|u'|^{m-2}u')dxdt \\
=\ &\int_S^T E^q\phi'\int_{|u'|\leq 1}\sigma(t)u(u'+|u'|^{m-2}u')dxdt \\
&+\int_S^T E^q\phi'\int_{|u'|>1}\sigma(t)u(u'+|u'|^{m-2}u')dx\,dt
\end{aligned}$$

We estimate the first term, we get

$$\begin{aligned}
&\left|\int_S^T E^q\phi'\int_{|u'|\leq 1}\sigma(t)u(u'+|u'|^{m-2}u')dxdt\right| \\
\leq\ &\int_s^t E^q\phi'\int_{|u'|\leq 1}\sigma(t)|uu'|dxdt + \int_S^T E^q\phi'\int_{|u'|\leq 1}\sigma(t)|uu'|\,|u'|^{m-2}dxdt \\
\leq\ &2\int_S^T E^q\phi'\int_{|u'|\leq 1}\sigma(t)|uu'|dxdt. \qquad (5)
\end{aligned}$$

Using the Hölder and Sobolev Poincaré inequalities and the energy identity from Lemma 4, we get

$$2\int_S^T E^q\phi'\int_{|u'|\leq 1}\sigma(t)|uu'|dxdt$$

$$\begin{aligned}
&\leq 2\int_S^T E^q\phi'\sigma(t)\left(\int_{|u'|\leq 1}|u|^p dx\right)^{\frac{1}{p}}\left(\int_{|u'|\leq 1}|u'|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} dt\\
&\leq 2\int_S^T E^q\phi'\sigma(t)\,\|u\|_{L^p}\left(\int_{|u'|\leq 1}|u'|^{\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}} dt\\
&\leq C(\Omega)\int_S^T E^{q+\frac{1}{p}}\phi'\sigma(t)\left[\int_{|u'|\leq 1}(u'g(u'))^{\frac{p}{2(p-1)}}dx\right]^{\frac{p-1}{p}} dt\\
&\leq C(\Omega)\int_S^T E^{q+\frac{1}{p}}\phi'\sigma(t)\left(\int_\Omega 1dx\right)^{\frac{p-2}{2(p-1)}}\left(\int_\Omega u'g(u')dx\right)^{\frac{1}{2}} dt\\
&\leq C(\Omega)\int_S^T E^{q+\frac{1}{p}}\phi'\sigma(t)\left(\frac{-E'}{\sigma(t)}\right)^{\frac{1}{2}} dt\\
&\leq C'(\Omega)\varepsilon\int_S^T E^{2(q+\frac{1}{p})}\phi' + C''(\Omega)\frac{1}{\varepsilon}\int_S^T(-E')dt\\
&\leq C'(\Omega)\varepsilon\int_s^t E^{2(q+\frac{1}{p})}\phi' dt + C''\frac{1}{\varepsilon}E(S). \qquad (6)
\end{aligned}$$

We choose q such that $2(q+\frac{1}{p}) = q+1$, thus we find $q=(p-2)/p$.
Using the Hölder inequality and the Sobolev imbedding, we obtain

$$\begin{aligned}
&\int_S^T E^q\phi'\int_{|u'|\geq 1}\sigma(t)ug(u')dxdt\\
&\leq \int_S^T E^q\phi'\sigma(t)\left(\int_\Omega |u|^m dx\right)^{\frac{1}{m}}\left(\int_{|u'|>1}|g(u')|^{\frac{m}{m-1}}\,dx\right)^{\frac{m-1}{m}} dt\\
&\leq c\int_S^T E^{q+\frac{1}{p}}\phi'^{\frac{1}{m}}(t)\left(\int_{|u'|>1}\sigma u'g(u')dx\right)^{\frac{m-1}{m}} dt\\
&\leq c\int_S^T E^{q+\frac{1}{p}}\phi'^{\frac{1}{m}}(t)(-E')^{\frac{m-1}{m}}dt.
\end{aligned}$$

Applying Young's inequality, we obtain

$$\begin{aligned}
&\int_S^T E^q\phi'\int_{|u'|\geq 1}\sigma(t)ug(u')\,dxdt\\
&\leq C(\Omega)\varepsilon_2^m\int_S^T\left(E^{q+\frac{1}{p}}\phi'^{\frac{1}{(m)}}(t)\right)^m dt + C(\Omega)\frac{1}{\varepsilon_2^{\frac{m}{m-1}}}\int_S^T(-E')\,dt\\
&\leq C(\Omega)\varepsilon_2^m\mu^m E^{\frac{(m-2)(p-1)}{p}}(0)\int_S^T E^{q+1}\phi' dt + C(\Omega)\frac{1}{\varepsilon_2^{\frac{m}{m-1}}}E(S). \qquad (7)
\end{aligned}$$

Set $\varepsilon_2 = \frac{\varepsilon''}{E(0)^{\frac{(m-2)(p-1)}{mp}}}$. Choosing ε and ε'' small enough, we deduce from (4), (6) and

(7) that

$$\int_S^T E(t)^{1+q}\phi'^{q+1} + C'^{q+\frac{1}{2}+\frac{1}{p}} + C''E(S) + C''''^{\frac{(m-2)(p-1)}{p(m-1)}}E(S)$$
$$\leq \left(\frac{C''^q + C'^{q+\frac{1}{p}-\frac{1}{2}} + C''''^{\frac{(m-2)(p-1)}{p(m-1)}}}{E(0)^q}\right)E(0)^q E(S)$$

where C, C', C'', C''', C'''' are different positive constants independent of $E(0)$. Hence, we deduce from Lemma 3 that

$$E(t) \leq \left(\frac{1+q}{q}\right)^{1/q}\left(C''^q + C'^{q+\frac{1}{p}-\frac{1}{2}} + C''''^{\frac{(m-2)(p-1)}{p(m-1)}1/q}\right)\left(\int_0^t \sigma(s)\,ds\right)^{-1/q}.$$

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