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A Simple Method For Estimating The Bounds Of Spectral Radius Of Nonnegative Irreducible Matrices*

Zhi Ming Yang†

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Abstract

Based on the Perron complement $P(A/A[\alpha])$ and generalized Perron complement $P_t(A/A[\alpha])$ of a nonnegative irreducible matrix A, we derive a simple and practical method that estimates the upper and lower bounds of the spectral radius of A in terms of norms of $A[\alpha]$ and its complements. Numerical examples show that this approach improves some of the classical estimates.

1 Introduction

The spectral theory of nonnegative matrix has a wide range of applications to Operations Research, Quantitative Economics, Graph theory and Markov chain theory. In 1989, in connection with a divide and conquer algorithm for computing the stationary distribution vector for a Markov chain, Meyer introduced, for an $n \times n$ nonnegative irreducible matrix A, the notion of the Perron complement. From then on, many applications involving the Perron complement have been emerging in the literature. In 2002, L.-Z. Lu introduced the generalized Perron complement. In this paper, we consider the problem of estimating the bounds of spectral radius of nonnegative irreducible matrix by using the concepts of Perron complement and the properties of matrix norm.

If $A = (a_{ij})$ is a nonnegative irreducible $n \times n$ matrix, then the Perron root $\rho(A)$ of A satisfies the classical inequalities of Frobenius [1, 2]:

$$\min_i r_i(A) = r(A) \leqslant \rho(A) \leqslant R(A) = \max_i r_i(A), \tag{1}$$

where $r_i(A) = \sum_{j=1}^n a_{ij} \, (i = 1, 2, ..., n)$.

In addition, a lot of estimates of $\rho(A)$ had been derived by Ledermann [6], Ostrowski [7], Brauer [8] and other authors. Though these estimates have improved the inequality (1) to a certain degree, many of them are very complicated. In order to introduce a simple method of estimation, we give the following notations:

For an arbitrary matrix $C = (c_{ij}) \in \mathrm{R}^{m\times n}$, let $\| C \|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} | c_{ij} |$, and $\| \cdot \|$ be a consistent matrix norm, and we know that

$$\| aC \| = |a| \cdot \| C \|, \quad \| C + D \| \le \| C \| + \| D \|, \tag{2}$$

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†Teachers College, Gansu Lianhe University, Lanzhou, Gansu, 730000, P. R. China

where $a \in \mathrm{R}$, $D \in \mathrm{R}^{m\times n}$.

Let α and β be nonempty ordered subsets of $\langle n\rangle := \{1, 2, ..., n\}$, both of strictly increasing integers and $\beta = \langle n\rangle \setminus \alpha$. For an $n \times n$ matrix A, $A[\alpha, \beta]$ will denote the submatrix of A whose rows and columns are determined by α and β respectively. In the special case that $\beta = \alpha$, $A[\alpha]$ will be used to denote $A[\alpha, \alpha]$, the principal submatrix of A corresponding to α.

DEFINITION 1.1 ([3, 4]). Let A be a nonnegative irreducible matrix of order n with the spectral radius $\rho(A)$, then the Perron complement of $A[\alpha]$ in A is defined as

$$P(A/A[\alpha]) = A[\beta] + A[\beta, \alpha](\rho(A)I - A[\alpha])^{-1}A[\alpha, \beta],$$

and the generalized Perron complement of $A[\alpha]$ in A is defined as

$$P_t(A/A[\alpha]) = A[\beta] + A[\beta, \alpha](tI - A[\alpha])^{-1}A[\alpha, \beta],$$

where $t > \rho(A[\alpha])$.

LEMMA 1.1 ([1]). If $\parallel A \parallel < 1$, then $I - A$ is nonsingular, and the inequality

$$\parallel (I - A)^{-1} \parallel \leq \frac{1}{1 - \parallel A \parallel}$$

holds.

LEMMA 1.2 ([3]). If A is a nonnegative irreducible matrix with the spectral radius $\rho(A)$, then the Perron complement $P(A/A[\alpha])$ is also a nonnegative irreducible matrix, and the spectral radius of $P(A/A[\alpha])$ is equivalent to $\rho(A)$.

LEMMA 1.3 ([1]). Let A be an $n \times n$ nonnegative irreducible matrix . Then A has a positive real eigenvalue equal to its spectral radius; To $\rho(A)$ there corresponds an eigenvector $x > 0$; And $\rho(A)$ is a simple eigenvalue of A .

LEMMA 1.4 ([4]). If A is a nonnegative irreducible matrix with the spectral radius $\rho(A)$, then for any $t > \rho(A[\alpha])$, $P_t(A/A[\alpha])$ is also a nonnegative irreducible matrix and

(1) if $t = \rho(A)$, then $\rho(P_t(A/A[\alpha])) = \rho(A)$;

(2) $\rho(P_t(A/A[\alpha]))$ is a strictly decreasing function of t.

2 Conclusions

For brevity in our discussion, we adopt the following notations: if $A \in \mathrm{R}^{n\times n}$, $\alpha \subset \langle n\rangle$, $\beta = \langle n\rangle \setminus \alpha$, then

$$A[\alpha] = A_\alpha\,,\ A[\alpha, \beta] = A_{\alpha\beta}\,,\ A[\beta, \alpha] = A_{\beta\alpha}\,,\ A[\beta] = A_\beta\,.$$

We may assume that $A = \begin{pmatrix} A_\alpha & A_{\alpha\beta} \\ A_{\beta\alpha} & A_\beta \end{pmatrix}$.

THEOREM 2.1. Let A be a nonnegative irreducible matrix. If $\rho(A) > \| A_\alpha \|$, then we have

$$\rho(A) \le \frac{1}{2}\Big[\big(\| A_\alpha \| + \| A_\beta \|\big) + \sqrt{\big(\| A_\alpha \| - \| A_\beta \|\big)^2 + 4 \| A_{\alpha\beta} \| \cdot \| A_{\beta\alpha} \|} \Big]. \tag{3}$$

PROOF. By Lemmas 1.2 and 1.3, we know that $\rho(A)$ is an eigenvalue of Perron complement $P(A/A_\alpha)$, and there exists a vector $x > 0$ such that

$$P(A/A_\alpha)\, x = [\, A_\beta + A_{\beta\alpha}(\rho(A)I - A_\alpha)^{-1} A_{\alpha\beta}\,]\, x = \rho(A)\, x,$$

By (2), it follows that

$$\rho(A) \cdot \| x \| \le \Big[\| A_\beta \| + \| A_{\beta\alpha} \| \cdot \| (\rho(A)I - A_\alpha)^{-1} \| \cdot \| A_{\alpha\beta} \| \Big] \cdot \| x \| \tag{4}$$

Since $\rho(A) > \| A_\alpha \|$, by Lemma 1.1 we know that $\rho(A)I - A_\alpha$ is nonsingular, and

$$\begin{aligned} \| (\rho(A)I - A_\alpha)^{-1} \| &= \frac{1}{\rho(A)} \Big\| (I - \frac{1}{\rho(A)} A_\alpha)^{-1} \Big\| \\ &\le \frac{1}{\rho(A)} \cdot \frac{1}{1 - \frac{1}{\rho(A)} \cdot \| A_\alpha \|} = \frac{1}{\rho(A) - \| A_\alpha \|}. \end{aligned}$$

Since $\| x \| > 0$, the inequality (4) will be

$$\rho(A) \le \| A_\beta \| + \frac{\| A_{\beta\alpha} \| \cdot \| A_{\alpha\beta} \|}{\rho(A) - \| A_\alpha \|}$$

Simplifying it, we obtain

$$\rho^2(A) - \big(\| A_\alpha \| + \| A_\beta \|\big)\, \rho(A) + \big(\| A_\alpha \| \cdot \| A_\beta \| - \| A_{\alpha\beta} \| \cdot \| A_{\beta\alpha} \|\big) \le 0.$$

So we have

$$\rho(A) \le \frac{1}{2}\Big[\big(\| A_\alpha \| + \| A_\beta \|\big) + \sqrt{\big(\| A_\alpha \| - \| A_\beta \|\big)^2 + 4 \| A_{\alpha\beta} \| \cdot \| A_{\beta\alpha} \|} \Big].$$

This completes the proof.

COROLLARY. Let A be a nonnegative irreducible matrix and

$$t_1 = \frac{1}{2}\Big[\big(\| A_\alpha \| + \| A_\beta \|\big) + \sqrt{\big(\| A_\alpha \| - \| A_\beta \|\big)^2 + 4 \| A_{\alpha\beta} \| \cdot \| A_{\beta\alpha} \|} \Big]. \tag{5}$$

if $\rho(A) > \| A_\alpha \|$, then $\rho(A) > \rho(P_{t_1}(A/A_\alpha))$.

PROOF. By Theorem 2.1 we know that $\rho(A) < t_1$, so according to Lemma 1.4 we obtain

$$\rho(A) = \rho(P_{\rho(A)}(A/A_\alpha)) > \rho(P_{t_1}(A/A_\alpha)).$$

Integrating the above two conclusions we now know that $\rho(A)$ satisfies

$$\rho(P_{t_1}(A/A_\alpha)) < \rho(A) < t_1\,,$$

where t_1 is just given as in (5).

3 Numerical Examples

In this section, we provide several examples.

EXAMPLE 3.1. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 1 & 1 \end{pmatrix}$.

Let $\alpha = \{1\}$. Then $\beta = \{2,3\}$ and

$$A_\alpha = 1, A_\beta = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}, A_{\beta\alpha} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, A_{\alpha\beta} = (1,2).$$

Let $\| \cdot \|$ be $\| \cdot \|_\infty$. Then it is clear that

$$\| A_\alpha \|_\infty = 1\,, \quad \| A_\beta \|_\infty = 6\,, \quad \| A_{\beta\alpha} \|_\infty = 4\,, \quad \| A_{\alpha\beta} \|_\infty = 3\,.$$

So by inequality (3), we will obtain an upper bound of $\rho(A)$, that is, $\rho(A) \leq \frac{1}{2}(7 + \sqrt{73}) = 7.7720$.

In addition, when $\alpha = \{1\}$, the generalized Perron complement of A_α will be

$$P_t(A/A_\alpha) = A_\beta + A_{\beta\alpha}(tI - A_\alpha)^{-1}A_{\alpha\beta} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 4 \\ 4 & 8 \end{pmatrix} \frac{1}{t-1}.$$

According to Theorem 2.1, we know that $t_1 = 7.7720$. So we have

$$\rho(A) > \rho(P_{t_1}(A/A_\alpha)) = 5.1923.$$

Similarly, we can get the upper bounds t_1 and lower bounds $\rho(P_{t_1}(A/A_\alpha))$ of $\rho(A)$ with different α listed as in Table 1.

Table 1:

α	$\rho(P_{t_1}(A/A_\alpha))$	t_1	F-Bound	L-Bound	O-Bound	B-Bound
{1}	5.1923	7.7720	8.0000	7.8661	7.6547	7.4642
{2}	5.3529	6.4495	$\vee$	$\vee$	$\vee$	$\vee$
{3}	5.2054	7.3589	$\rho(A)$	$\rho(A)$	$\rho(A)$	$\rho(A)$
{1,2}	3.7198	7.3589	$\vee$	$\vee$	$\vee$	$\vee$
{1,3}	4.9933	6.4495	4.0000	4.1547	4.5275	4.8284
{2,3}	3.3100	7.7720				

Here F-Bound, L-Bound, O-Bound and B-Bound denote the Frobenius', Ledermann's, Ostrowski's and Brauer's bounds respectively (See [5]).

From the table we see that the bounds obtained by our method especially under the case of $\alpha = \{2\}$ are much better than those bounds obtained by other methods (in fact, $\rho(A) = 5.74165738\cdots$). And for other values of α, such as $\alpha = \{1, 2\}$ and $\alpha = \{2, 3\}$, though the upper bounds have been improved to a certain degree, the lower bounds are far from $\rho(A)$. In fact, as for our algorithms introduced here, the rows which have the maximal row sum and the minimal row sum play a decisive role. In general, if the minimal row sum is close to the maximal row sum, we can take the row which has the maximal row sum as α; otherwise, if the minimal row sum is much smaller than the maximal row sum, we take the row which has the minimal row sum as α.

EXAMPLE 3.2. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$.

We list the upper bounds t_1 and lower bounds $\rho(P_{t_1}(A/A_\alpha))$ of $\rho(A)$ obtained by different α as in Table 2.

Table 2:

α	$\rho(P_{t_1}(A/A_\alpha))$	t_1	F-Bound	L-Bound	O-Bound	B-Bound
{1}	17.1608	20.2596	21.0000	20.9759	20.5000	20.2596
{6}	16.1632	20.2596	$\vee$	$\vee$	$\vee$	$\vee$
{1,2}	17.0279	19.4582	$\rho(A)$	$\rho(A)$	$\rho(A)$	$\rho(A)$
{1,2,3}	16.7430	19.1168	$\vee$	$\vee$	$\vee$	$\vee$
{4,5,6}	12.8470	19.1168	6.0000	6.0247	7	7.8990

We see from Table 2 that the bounds obtained for $\alpha = \{1\}$ or $\alpha = \{1, 2\}$ taken from the row having the minimal row sum are better than those for $\alpha = \{6\}$ taken from the row having the maximal row sum. Finally, we have $17.1608 \leq \rho(A) \leq 19.1168$ ($\rho(A) = 17.2069$). This result is better than other estimates given by Frobenius, Ledermann, Ostrowski and Brauer. Despite that, the upper bound is still not better than the result obtained by Lu [4], that is, $15.6944 \leq \rho(A) \leq 18.0498$, where the upper bound is the twice iteration of the Brauer's upper bound. As we can see from Table 2 that it is difficult to get a better upper bound just by selecting a proper α, but if we continue to apply the method of this paper to the Perron complements $P_{t_1}(A/A_\alpha)$, we will obtain some closer upper bounds of $\rho(A)$. For example, when we take $\alpha = \{1, 2\}$, we have

$$P_{t_1}(A/A_\alpha) = \begin{pmatrix} 3.2966 & 3.2966 & 3.2966 & 3.2966 \\ 3.2966 & 4.2966 & 4.2966 & 4.2966 \\ 3.2966 & 4.2966 & 5.2966 & 5.2966 \\ 3.2966 & 4.2966 & 5.2966 & 6.2966 \end{pmatrix} \triangleq B.$$

Then, we take $\langle n' \rangle := \{1, 2, 3, 4\}, \alpha' = \{1\}$, $\beta' = \langle n' \rangle \setminus \alpha' = \{2, 3, 4\}$.

By (5), we obtain $t_1 \leq 17.4455$, where we replace A, α, β with B, α', β' respectively. Thus we have

$$17.1608 \leq \rho(A) \leq 17.4455.$$

These are very good lower and upper bounds for $\rho(A)$. It shows that Lu's result has been greatly improved.

The method derived here may provide better estimates for the bounds of the spectral radius, and is simple and practical.

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