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Integrability Of Cosine Trigonometric Series With Coefficients Of Bounded Variation*

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Abstract

In this paper condition of integrability of cosine trigonometric series with coefficients of bounded variation of order p is obtained. The results generalize some previous results of Telyakovskiĭ.

1 Introduction

In the literature, there are many studies related to the trigonometric series, and particularly the cosine series. We refer to the excellent monograph by R. P. Boas, Jr. [1].

The first results pertaining to the series

$$\frac{a_0}{2}+\sum_{k=1}^{\infty}a_k\cos kx \tag{1}$$

considered the case of monotone coefficients. Later, some authors investigated the series (1) with quasi-monotone coefficients ($a_{n+1}\leq a_n(1+\alpha/n)$, $n\geq n_0$, $\alpha>0$).

Several papers have been written on the series (1) when the sequence $\{a_k\}$ is a null-sequence and convex or quasi-convex, i.e. $\triangle^2 a_k\geq 0$ or

$$\sum_{k=1}^{\infty}(k+1)|\triangle^2 a_k|<\infty, \tag{2}$$

where $\triangle^2 a_k=\triangle\left(\triangle a_k\right)$, $\triangle a_k=a_k-a_{k+1}$.

Furthermore, when $\{a_k\}$ is a null-sequence of bounded variation, i.e.

$$\sum_{k=1}^{\infty}|\triangle a_k|<\infty,$$

is also considered.

We shall consider the series (1) whose coefficients tend to zero and satisfy any condition that provides the convergence of the series (1) on $(0,\pi]$. Let us denote its

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sum by $f(x)$. If the coefficients a_k are quasi-convex, it is well-known that f is an integrable function on $[0,\pi]$ (see for example [2], page 264), and the following estimate is valid

$$\int_0^\pi |f(x)|dx \le \pi \sum_{k=1}^{\infty}(k+1)|\triangle^2 a_k|.$$

In a similar direction, among others, Telyakovskiĭ [2] obtained some estimates of the integrals of the following form

$$\int_{\pi/(m+1)}^{\pi/\ell} |f(x)|dx, \quad 1 \le \ell \le m, \quad (\ell, m \in \mathbb{N}), \tag{3}$$

expressed in terms of the coefficients a_k, where he used null-sequences of bounded variation of second order ($\sum_{k=1}^{\infty} |\triangle^2 a_k| < \infty$), instead of quasi-convex null-sequences.

It is obvious that the condition

$$\sum_{k=1}^{\infty} |\triangle^2 a_k| < \infty \tag{4}$$

is a weaker condition than the condition (2).

The following definition is introduced in [4]: A sequence $\{a_k\}$ is of bounded variation of integer order $p \ge 0$ if

$$\sum_{k=1}^{\infty} |\triangle^p a_k| < \infty, \tag{5}$$

where $\triangle^p a_k = \triangle\left(\triangle^{p-1} a_k\right) = \triangle^{p-1} a_k - \triangle^{p-1} a_{k+1}$, and we agree with $\triangle^0 a_k = a_k$.

In [4] an example is given to show that (5) is an effective generalization of the null sequences of bounded variation. This fact motivates the author to consider the series (1) with coefficients that satisfy the condition (5).

The main goal of the present note is to use (5), $p \ge 2$, instead of (4), to prove some estimates of the form (3) that shall generalize some results of Telyakovskiĭ in [2].

We write $g(u) = O_p\left(h(u)\right)$, $u \to 0$, if there exists a positive constant A_p, that depends only on p, such that $g(u) \le A_p h(u)$ in a neighborhood of the point $u = 0$. The constant A_p may be, in general, different in different estimates.

2 Main Results

We need the following notations

$$\begin{aligned}
B_0^1(x) &= \frac{1}{2}, \\
B_k^1(x) &= \frac{1}{2} + \cos x + \cdots + \cos kx \quad \text{for} \quad k \ge 1, \\
B_k^p(x) &= \sum_{\nu=0}^{k} B_\nu^{p-1}(x) \quad \text{for} \quad p = 2, 3, \ldots \ \text{ and } \ k \ge 0,
\end{aligned}$$

and inequalities (see [3], page 20):

(i) $B_k^p(x) \ge 0, \qquad \forall p \ge 2, \quad -\pi \le x \le \pi;$

(ii) $B_k^p(x) = O\left((k+1)^p\right), \qquad \forall p \ge 1, \quad -\pi \le x \le \pi;$

(iii) $B_k^p(x) = O\left(\frac{1}{x^p}\right), \qquad \forall p \ge 1, \quad 0 < x \le \pi.$

THEOREM 1. If $a_k \to 0$ as $k \to \infty$ and (5) is satisfied, then the series (1) converges on $(0, \pi]$, uniformly on $[\varepsilon, \pi]$, for every $\varepsilon > 0$, and for $p \ge 2$, $1 \le \ell \le m$, the sum function $f(x)$ satisfies

$$\begin{aligned}\int_{\pi/(m+1)}^{\pi/\ell} |f(x)|dx \quad &= \quad O_p\left(\frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{(k+1)^{p-1}}{\ell}|\triangle^{p-1}a_k|\right)\\ &+O_p\left(\sum_{k=\ell}^{\infty}\min(k+1-\ell,\, m+1-\ell)(k+1)^{p-2}|\triangle^p a_k|\right). \quad (6)\end{aligned}$$

PROOF. Applying Abel's transformation p times we get that

$$\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx = \sum_{k=0}^{n-p}\triangle^p a_k B_k^p(x) + \sum_{j=0}^{p-1}\triangle^j a_{n-j} B_{n-j}^{j+1}(x).$$

In view of the hypotheses of our Theorem and $B_k^p(x) = O\left(\frac{1}{x^p}\right)$, $0 < x \le \pi$, it is obvious that the series (1) converges uniformly on $[\varepsilon, \pi]$, $\varepsilon > 0$, and the following representation holds

$$f(x) = \sum_{k=0}^{\infty}\triangle^p a_k B_k^p(x). \tag{7}$$

Let i be a positive integer and $x \in \left(\frac{\pi}{i+1}, \frac{\pi}{i}\right]$. Using the equality

$$\sum_{k=0}^{i-1}\triangle^p a_k B_k^p(x) = \sum_{k=0}^{i-1}\triangle^{p-1} a_k B_k^{p-1}(x) - \triangle^{p-1} a_i B_{i-1}^p(x),$$

from (7) we have

$$f(x) = \sum_{k=0}^{i-1}\triangle^{p-1} a_k B_k^{p-1}(x) + \sum_{k=i}^{\infty}\triangle^p a_k\left[B_k^p(x) - B_{i-1}^p(x)\right].$$

Since $|B_k^{p-1}(x)| = O_p\left((k+1)^{p-1}\right)$ and

$$|B_k^p(x) - B_{i-1}^p(x)| = O_p\left(\frac{1}{x^p}\right),$$

we have

$$\int_{\pi/(i+1)}^{\pi/i}|f(x)|dx = O_p\left(\sum_{k=0}^{i-1}|\triangle^{p-1}a_k|\frac{(k+1)^{p-1}}{i(i+1)} + (i+1)^{p-2}\sum_{k=i}^{\infty}|\triangle^p a_k|\right).$$

Thus

$$\int_{\pi/(m+1)}^{\pi/\ell} |f(x)|dx = O_p\left(\sum_{i=\ell}^{m}\sum_{k=0}^{i-1} |\triangle^{p-1}a_k| \frac{(k+1)^{p-1}}{i(i+1)} + \sum_{i=\ell}^{m}\sum_{k=i}^{\infty}(k+1)^{p-2}|\triangle^p a_k|\right). \tag{8}$$

For the first term in the parentheses of the right-hand side of (8) we have

$$\begin{aligned}
&\sum_{i=\ell}^{m}\sum_{k=0}^{i-1} |\triangle^{p-1}a_k| \frac{(k+1)^{p-1}}{i(i+1)} \\
=\ & \sum_{i=\ell}^{m}\sum_{k=0}^{\ell-1} |\triangle^{p-1}a_k| \frac{(k+1)^{p-1}}{i(i+1)} + \sum_{i=\ell+1}^{m}\sum_{k=\ell}^{i-1} |\triangle^{p-1}a_k| \frac{(k+1)^{p-1}}{i(i+1)} \\
=\ & \sum_{k=0}^{\ell-1}(k+1)^{p-1}|\triangle^{p-1}a_k|\left(\frac{1}{\ell} - \frac{1}{m+1}\right) \\
& \quad + \sum_{k=\ell}^{m-1}(k+1)^{p-1}|\triangle^{p-1}a_k|\left(\frac{1}{k+1} - \frac{1}{m+1}\right) \\
\leq\ & \frac{m+1-\ell}{m}\sum_{k=0}^{\ell-1}\frac{(k+1)^{p-1}}{\ell}|\triangle^{p-1}a_k| + \sum_{k=\ell}^{m}\sum_{j=k}^{\infty}(j+1)^{p-2}|\triangle^p a_j|.
\end{aligned} \tag{9}$$

Finally, the last term in (9) can be written as

$$\begin{aligned}
\sum_{i=\ell}^{m}\sum_{k=i}^{\infty}(k+1)^{p-2}|\triangle^p a_k| &= \sum_{i=\ell}^{m}\sum_{k=i}^{m}(k+1)^{p-2}|\triangle^p a_k| + \sum_{i=\ell}^{m}\sum_{k=m+1}^{\infty}(k+1)^{p-2}|\triangle^p a_k| \\
&= \sum_{k=\ell}^{m}(k+1-\ell)(k+1)^{p-2}|\triangle^p a_k| \\
&\quad +(m+1-\ell)\sum_{k=m+1}^{\infty}(k+1)^{p-2}|\triangle^p a_k|.
\end{aligned} \tag{10}$$

The proof of theorem follows from (8), (9) and (10).

Now we estimate the integral in (6) only in terms of p-th order difference of the sequence $\{a_k\}$.

COROLLARY 1. If the coefficients of the series (1) satisfy conditions of the Theorem 1, then

$$\int_{\pi/(m+1)}^{\pi/\ell} |f(x)|dx = O_p\left(\frac{m+1-\ell}{m}\sum_{k=0}^{\infty}\min\left(\frac{(k+1)^2}{\ell}, k+1, m\right)(k+1)^{p-2}|\triangle^p a_k|\right),$$

holds, where $1 \leq \ell \leq m$, $p \geq 2$.

PROOF. To deduce from (6) the required relation, we use the identity

$$\triangle^{p-1}a_k = \sum_{i=k}^{\infty}(\triangle^p a_i).$$

We have

$$
\begin{aligned}
& \sum_{k=0}^{\ell-1} \frac{(k+1)^{p-1}}{\ell} |\triangle^{p-1} a_k| \\
\leq & \sum_{k=0}^{\ell-1} \frac{(k+1)^{p-1}}{\ell} \sum_{i=k}^{\infty} |\triangle^{p} a_i| \\
= & \sum_{i=0}^{\ell-1} \sum_{k=0}^{i} \frac{(k+1)^{p-1}}{\ell} |\triangle^{p} a_i| + \sum_{i=\ell}^{\infty} \sum_{k=0}^{\ell-1} \frac{(k+1)^{p-1}}{\ell} |\triangle^{p} a_i| \\
\leq & \sum_{i=0}^{\ell-1} \frac{(i+1)^{p}}{\ell} |\triangle^{p} a_i| + \ell^{p-1} \sum_{i=\ell}^{\infty} |\triangle^{p} a_i| \\
\leq & \sum_{i=0}^{\ell-1} \frac{(i+1)^{2}}{\ell} (i+1)^{p-2} |\triangle^{p} a_i| + \ell \sum_{i=\ell}^{\infty} (i+1)^{p-2} |\triangle^{p} a_i|. \qquad (11)
\end{aligned}
$$

If $k < m$, then we can estimate the second term in (6) by means of the fact that

$$k+1-\ell \leq k+1-\ell \frac{k+1}{m} = \frac{m-\ell}{m}(k+1).$$

From the above and (11) along with (6) we obtain the required estimate.

The following Corollaries 2 and 3 are immediate from Theorem 1 and Corollary 1, respectively.

COROLLARY 2. ([2]) If $a_k \to 0$ as $k \to \infty$ and (4) holds, then the series (1) converges on $(0, \pi]$, uniformly on $[\varepsilon, \pi]$, for every $\varepsilon > 0$, and for $1 \leq \ell \leq m$, f satisfies

$$
\begin{aligned}
\int_{\pi/(m+1)}^{\pi/\ell} |f(x)| dx &= O\left(\frac{m+1-\ell}{m} \sum_{k=0}^{\ell-1} \frac{k+1}{\ell} |\triangle a_k| \right) \\
&+ O\left(\sum_{k=\ell}^{\infty} \min(k+1-\ell, m+1-\ell) |\triangle^2 a_k| \right).
\end{aligned}
$$

COROLLARY 3. ([2]) If the coefficient sequence of the series (1) tends to zero and satisfies condition (4), then the following estimate

$$\int_{\pi/(m+1)}^{\pi/\ell} |f(x)| dx = O\left(\frac{m+1-\ell}{m} \sum_{k=0}^{\infty} \min\left(\frac{(k+1)^2}{\ell}, k+1, m \right) |\triangle^2 a_k| \right),$$

holds, $1 \leq \ell \leq m$.

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