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Analysis Of Adomian Series Solution To A class Of Nonlinear Ordinary Systems Of Raman Type*

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Abstract

In this paper the Adomian decomposition method is used to find an analytic solution for nonlinear reaction system of Raman type. In this approach, the solutions are found in the form of a convergent power series with easily computed components. Convergence analysis of Adomian series solution for a class of these type of nonlinear ODEs is discussed and a numerical example is presented.

1 Introduction

Raman equations usually account for a large number of effects but the major interactions are the attenuation, and the power transfer between waves. The steady-state Raman amplified system can be described by a set of coupled nonlinear equations (see [1])

$$\pm\frac{dP_i}{dz} = -\alpha_i P_i + \sum_{j=1}^{i-1} C_{ij} P_j P_i - \sum_{j=i+1}^{m} \frac{\nu_i}{\nu_j} C_{ji} P_j P_i. \tag{1}$$

with initial condition

$$P(0) = [P_1(0), \dots, P_m(0)]^T \tag{2}$$

and where $C_{ij} = \frac{g_r(\nu_j - \nu_i)}{\Gamma A_{eff}}, C_{ji} = \frac{g_r(\nu_i - \nu_j)}{\Gamma A_{eff}}$ and $\alpha_i > 0$, $i = 1, \dots, m$.

The nonlinear term is of the type $N(P_i, P_j) \equiv f(P_i, P_j) = P_i P_j$ has Adomian polynomials representation $f(P_i, P_j) = \sum_{n=0}^{\infty} A_n$ where the formula of A_n is given by (6).

The $\pm$ signs stand for forward and backward waves, respectively. P_i, ν_i and α_i are the power, frequency and attenuation coefficient of the i-th wave, respectively. A_{eff} is the optical fibre effective area, the factor Γ accounts the polarization random effects. $g_r(\nu_j - \nu_i)$ is the Raman gain coefficient from wave j to wave i. The frequencies ν_i are numbered in decreasing order ($i = 1, 2, ..., m$). In optical fiber, due to the amorphous nature of Silica, the Raman gain coefficient presents a fairly broad shape.

The Adomian decomposition method (ADM) has been used to solve effectively and accurately a large set of differential equations (see, for instance [2]) as linear or nonlinear, ordinary or partial, deterministic or stochastic. In this method, the solution

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is presented as the sum of an infinite series, rapidly converging to an accurate solution. In particular, it is quite effective for dealing with nonlinear problems and do not involve linearization of the problem. Evans and Raslan [3] applied a decomposition method for solving delay differential equation. In [4], Gu and Li introduced a modified ADM to solve a class of systems of nonlinear differential equations. The method can be mechanized in Maple and a procedure is written to solve the approximate analytic solution of the systems. Approximate analytic solution for nonlinear reaction diffusion system of Lotka-Volterra type were obtained by Alabdullatif, Abdusalam and Fahmy [5]. Ibrahim L. El-Kalla [6] introduces a new formula for Adomian polynomials. Based on this new formula, error analysis of Adomian series solution for a class of nonlinear differential equations is discussed. Afrouzi and Khademloo applied the ADM to a quasilinear parabolic equation. In general, the large majority of papers about ADM, just applied the method without studding its convergence, some exception is [6].

In the scope of these type of nonlinear ODEs models (1), the main contribution of this paper is to use the Adomian decomposition method to solve the Raman propagation equations. The solutions are found in the form of a convergent power series and the convergence analysis of Adomain series solution for a class of these type of ODEs is discussed.

The paper is organized as follows: In section 2, we review the Adomian decomposition method. In section 3, we present the analysis of the Adomian decomposition method applied to nonlinear coupled system. Section 4 is devoted to the study the convergence of the method and estimate the maximum absolute error of the truncated series. Finally, in section 5 a numerical example is presented.

2 A Brief Review of ADM

For the purpose of illustration of the methodology to the proposed method, using ADM, we consider the general form of equation,

$$Lu + Gu + Nu = 0, \tag{3}$$

where u is the unknown function, L represents a linear operator which is easily invertible, G is a linear operator and Nu represents the nonlinear term. We assume that the operator L is invertible and it can be taken as the define integral with respect to z from z_0 to z, i.e., $L^{-1} = \int_{z_0}^{z}(.)d\tilde{z}$. Applying the inverse operator L^{-1} to both sides of (3) and using the initial condition, i.e., $u(0) = g(z)$ we find

$$u = g - L^{-1}[Gu + Nu]. \tag{4}$$

The nonlinear term Nu can be decomposed by an infinite series of polynomials given by

$$Nu = \sum_{n=0}^{\infty} A_n(u_0, u_1, \ldots, u_n), \tag{5}$$

where $A_n(u_0, u_1, \ldots, u_n)$ are the appropriate Adomian's polynomials and are defined by

$$n!A_n = \frac{d^n}{d\lambda^n}\left[f\left(\sum_{k=0}^{\infty}\lambda^k u^k\right)\right]_{\lambda=0}, n \geq 0. \quad (6)$$

The ADM assumes a series that the unknown function $u(z)$ can be expressed by a infinite series of the form

$$u(z) = \sum_{n=0}^{\infty} u_n(z). \quad (7)$$

Taking into account (5) and substituting (7) into (4), we obtain

$$\sum_{n=0}^{\infty} u_n(z) = g - L^{-1}\left[G\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right]. \quad (8)$$

Identifying the zero component u_0, the remaining components can be determined by using recurrence relation, i.e.,

$$u_0 = g(z), \quad u_{n+1} = -L^{-1}[G(u_n) + A_n], \quad n > 0. \quad (9)$$

The scheme (9) can easily determine the components $u_n(z)$. It is in principle, possible to calculate more components in the decomposition series to enhance the approximation. Consequently, we can recursively determine every term of the series $\sum_{n=0}^{\infty} u_n(z)$ and hence the solution $u(z)$ is readily obtained, i.e., using the above recursive relationship, we construct the solutions $u(z)$ as $u(z) = \lim_{n\to\infty} S_n$, where $S_n = \sum_{i=0}^{n} u_i(z)$ for $n \geq 0$.

It is interesting to note that, we obtain the solution by using the initial condition only.

3 The Analysis of the ADM

For simplicity, we are interested to deal with Adomian decomposition solution associated with the operator L^{-1}.

Following Adomian decomposition method [2], the system (1) can be written in an operator form as

$$\pm LP_i = -\alpha_i P_i + \sum_{j=1}^{i-1} C_{ij} N(P_j, P_i) - \sum_{j=i+1}^{m} \frac{\nu_i}{\nu_j} C_{ji} N(P_j, P_i), \quad (10)$$

where $L = \frac{\partial}{\partial z}$.

Operating with L^{-1} on both sides of (10) and using the initial conditions (2), we get

$$\pm P_i = P_i(0) + L^{-1}\left[-\alpha_i P_i + \sum_{j=1}^{i-1} C_{ij} N(P_j, P_i) - \sum_{j=i+1}^{m} \frac{\nu_i}{\nu_j} C_{ji} N(P_j, P_i)\right]. \quad (11)$$

The Adomian decomposition method assumes a series solution of the unknown functions $P_i(z), \quad i = 1, \dots, m$, are given by

$$P_i(z) = \sum_{n=0}^{\infty} P_{i,n}(z). \tag{12}$$

Substituting (12) into (11) and taking into account that

$$N(P_j, P_i) = \sum_{n=0}^{\infty} A_n(P_j P_i), \tag{13}$$

we obtain

$$\sum_{n=0}^{\infty} P_{in}(z) = P_i(0) + \left[-\alpha_i \sum_{n=0}^{\infty} P_{i,n} + \sum_{j=1}^{i-1} C_{ij} \sum_{n=0}^{\infty} A_n - \sum_{j=i+1}^{m} \frac{\nu_i}{\nu_j} C_{ji} \sum_{n=0}^{\infty} A_n \right].$$

Given the components $P_{i,0}$, the remaining components $P_{i,n+1}$, $n \geq 0$, can be completely determined using the previous terms, i.e.,

$$P_{i,n+1} = L^{-1} \left[-\alpha_i P_{i,n} + \sum_{j=1}^{i-1} C_{ij} A_n - \sum_{j=i+1}^{m} \frac{\nu_i}{\nu_j} C_{ji} A_n \right].$$

Hence, the series solutions is entirely evaluated.

In the following lemma we obtain an explicit formula for A_n.

LEMMA 1. If the polynomials A_n for $N(P_j, P_i)$ are given by

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N \left(\sum_{k=0}^{n} \lambda^k P_{j,k}, \sum_{k=0}^{n} \lambda^k P_{i,k} \right) \right] \Bigg|_{\lambda=0} \tag{14}$$

then

$$A_n = \sum_{k=0}^{n} P_{j,k} P_{i,n-k}. \tag{15}$$

PROOF. First, we observe that

$$\begin{aligned}
& \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} N \left(\sum_{k=0}^{n} \lambda^k P_{j,k}, \sum_{k=0}^{n} \lambda^k P_{i,k} \right) \right] \\
= \; & \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\sum_{k=0}^{n} \lambda^k P_{j,k} \sum_{k=0}^{n} \lambda^k P_{i,k} \right) \right] \\
= \; & \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(P_{j,0} \sum_{k=0}^{n} \lambda^k P_{i,k} + \lambda P_{j,1} \sum_{k=0}^{n} \lambda^k P_{i,k} + \dots + \lambda^n P_{j,n} \sum_{k=0}^{n} \lambda^k P_{i,k} \right) \right] \\
= \; & \frac{1}{n!} [n! P_{j,0} P_{i,n} + (n+1) n \cdots 2 \lambda P_{j,1} P_{i,n} + n! P_{j,1} P_{i,n-1}
\end{aligned}$$

$$+(n+2)(n+1)n\cdots 3\lambda^2 P_{j,2}P_{i,n} + (n+1)n\cdots 2\lambda P_{j,2}P_{i,n-1}$$
$$+n!P_{j,2}P_{i,n-2} + \cdots + (2n)(2n-1)\cdots(n+1)\lambda^n P_{j,n}P_{i,n} + \cdots$$
$$+(n+1)n(n-1)\cdots 2\lambda P_{j,n}P_{i,1} + n!P_{j,n}P_{i,0}].$$

Hence, taking into account (14) and the previous relation, we obtain

$$\begin{aligned} A_n &= \frac{1}{n!}\left[\frac{d^n}{d\lambda^n} N\left(\sum_{k=0}^{n}\lambda^k P_{j,k}, \sum_{k=0}^{n}\lambda^k P_{i,k}\right)\right]\Bigg|_{\lambda=0} \\ &= P_{j,0}P_{i,n} + P_{j,1}P_{i,n-1} + P_{j,2}P_{i,n-2} + \cdots + P_{j,n}P_{i,0}, \end{aligned}$$

which is our result.

4 Convergence Analysis

In this section a condition that guarantees the existence of a unique solution is introduced in Proposition 1, the convergence of the series solution (11) is proved in Proposition 2, and the maximum absolute error of the truncated series (11) is estimated in Proposition 3.

Denote by $Y = (C[I], \|.\|)$ the Banach space of all continuous functions on $I = [0,T] \subseteq \mathbb{R}$ with the norm $\|P\| = \sum_{i=1}^{m} \max_{z\in I}|P_i|$, and where $\|P_i\| = \max_{z\in I}|P_i(z)|$ for $i = 1,\ldots,m$.

In the following Proposition, we discuss the existence and uniqueness of the solution of the problem (2) and (11).

PROPOSITION 1. Suppose that the following conditions hold:

($\mathcal{H}_1$) There is a constant $M_5 > 0$ such that $\|P_i\| < \frac{1}{2}M_5$ for all $i = 1,\ldots,m$ (which makes sense since signals are bounded).

($\mathcal{H}_2$) Let $I = [0,T] \subseteq \mathbb{R}$, M_0 be the smallest positive real number that satisfies $|P_k(0) - Q_k(0)| \leq M_0 \max_{z\in I}|P_k(z) - Q_k(z)|$, and

$$M_6 = \max_{1\leq k\leq m}\left\{\alpha_k, M_5\sum_{j=1}^{k-1} C_{kj}, M_5\sum_{j=k+1}^{m}\frac{\nu_k}{\nu_j}C_{jk}\right\}.$$

such that if $\gamma = m(M_0 + M_6 T)$ then $0 < \gamma < 1$.

Then the problem (2) and (11) has a unique solution.

PROOF. First, note that hypothesis $\mathcal{H}_1$ implies that

$$\begin{aligned} \|P_jP_k - Q_jQ_k\| &= \|P_jP_k - P_kQ_j + P_kQ_j - Q_jQ_k\| \\ &= \|P_k(P_j - Q_j) + Q_j(P_k - Q_k)\| \\ &\leq \|P_k\|\,\|P_j - Q_j\| + \|Q_j\|\,\|P_k - Q_k\| \\ &\leq \frac{1}{2}M_5\|P_j - Q_j\| + \frac{1}{2}M_5\|P_k - Q_k\| \leq M_5\|P_k - Q_k\|. \end{aligned}$$

We define a mapping $F : Y \to Y$, where $P \mapsto FP(z)$ and $FP(z) = (F_1P_1(z), \dots, F_mP_m(z))$ with

$$F_iP_i(z) = P_i(0) + L^{-1}\left[-\alpha_i P_i + \sum_{j=1}^{i-1} C_{ij}P_jP_i - \sum_{j=i+1}^{m} \frac{\nu_i}{\nu_j} C_{ji}P_jP_i\right].$$

Here, properties such as continuity will be understood in a componentwise manner.

Let $P, Q \in Y$. We have

$$\begin{aligned}
\|FP - FQ\| &= \sum_{i=1}^{m} \max_{z\in I} |F_iP_i - F_iQ_i| \le m \max_{z\in I} |F_kP_k(z) - F_kQ_k(z)| \\
&= m \max_{z\in I}\left|\left(P_k(0) + L^{-1}\left[-\alpha_k P_k + \sum_{j=1}^{k-1} C_{kj}P_jP_k - \sum_{j=k+1}^{m} \frac{\nu_k}{\nu_j} C_{jk}P_jP_k\right]\right)\right. \\
&\quad \left. - \left(Q_k(0) + L^{-1}\left[-\alpha_k Q_k + \sum_{j=1}^{k-1} C_{kj}Q_jQ_k - \sum_{j=k+1}^{m} \frac{\nu_k}{\nu_j} C_{jk}Q_jQ_k\right]\right)\right| \\
&\le m|P_k(0) - Q_k(0)| + m \max_{z\in I} \left|L^{-1}\left[-\alpha_k P_k + \alpha_k Q_k\right]\right| \\
&\quad + m \max_{z\in I} \left|L^{-1}\left[\sum_{j=1}^{k-1} C_{kj}P_jP_k - \sum_{j=1}^{k-1} C_{kj}Q_jQ_k\right]\right| \\
&\quad + m \max_{z\in I} \left|L^{-1}\left[-\sum_{j=k+1}^{m} \frac{\nu_k}{\nu_j} C_{jk}P_jP_k + \sum_{j=k+1}^{m} \frac{\nu_k}{\nu_j} C_{jk}Q_jQ_k\right]\right| \\
&\le m|P_k(0) - Q_k(0)| + m\alpha_k \max_{z\in I} L^{-1}|Q_k - P_k| \\
&\quad + m \sum_{j=1}^{k-1} C_{kj} \max_{z\in I} L^{-1}|P_jP_k - Q_jQ_k| \\
&\quad + m \sum_{j=k+1}^{m} \frac{\nu_k}{\nu_j} C_{jk} \max_{z\in I} L^{-1}|Q_jQ_k - P_jP_k| \\
&\le m|P_k(0) - Q_k(0)| + m\alpha_k \max_{z\in I} L^{-1}|Q_k - P_k| \\
&\quad + mM_5 \sum_{j=1}^{k-1} C_{kj} \max_{z\in I} L^{-1}|P_k - Q_k| \\
&\quad + mM_5 \sum_{j=k+1}^{m} \frac{\nu_k}{\nu_j} C_{jk} \max_{z\in I} L^{-1}|P_k - Q_k| \\
&\le m|P_k(0) - Q_k(0)| + m\alpha_k \max_{z\in I} |Q_k - P_k| \max_{z\in I} \int_0^z d\tilde{z}
\end{aligned}$$

$$
\begin{aligned}
&+mM_5 \sum_{j=1}^{k-1} C_{kj} \max_{z\in I} |P_k - Q_k| \max_{z\in I} \int_0^z d\tilde{z} \\
&+mM_5 \sum_{j=k+1}^{m} \frac{\nu_k}{\nu_j} C_{jk} \max_{z\in I} |P_k - Q_k| \max_{z\in I} \int_0^z d\tilde{z} \\
\leq \;& m|P_k(0) - Q_k(0)| + mM_6 \max_{z\in I} |P_k(z) - Q_k(z)| \max_{z\in I} \int_0^z d\tilde{z} \\
= \;& m|P_k(0) - Q_k(0)| + mM_6 \max_{z\in I} |P_k(z) - Q_k(z)| T \\
\leq \;& m(M_0 + M_6 T) \max_{z\in I} |P_k(z) - Q_k(z)| \\
\leq \;& m(M_0 + M_6 T) \|P_k - Q_k\| \\
= \;& \gamma \|P_k - Q_k\|.
\end{aligned}
$$

Under the condition $0 < \gamma < 1$ the mapping F is a contraction. Therefore, by the Banach fixed-point theorem, there exists a unique solution to problem (2) and (11), which completes the proof.

PROPOSITION 2. Suppose that hypotheses $(\mathcal{H}_1)$-$(\mathcal{H}_2)$ hold together with:

$(\mathcal{H}_3)$ Let $M_7 = \max_{1\leq k\leq m} \left\{\alpha_k, \sum_{j=1}^{k-1} C_{kj}, \sum_{j=k+1}^{m} \frac{\nu_k}{\nu_j} C_{jk}\right\}$ and $\bar{\gamma} = M_7 T$ such that $0 < \bar{\gamma} < 1$.

Then the series solution (12) of problem (2) and (11), using ADM, converges.

PROOF. Let S_n and S_q be arbitrary partial sums with $n \geq q$. We are going to prove that $\{S_n\}$ is a Cauchy sequence in Y. Indeed,

$$
\begin{aligned}
\|S_n - S_q\| &= \max_{z\in[0,T]} |S_n - S_q| = \max_{z\in[0,T]} \left| \sum_{k=q+1}^{n} P_{i,k} \right| \\
&= \max_{z\in[0,T]} \left| \sum_{k=q+1}^{n} L^{-1} \left[-\alpha_{k-1} P_{i,k-1} + \sum_{j=1}^{(k-1)-1} C_{k-1j} A_{k-1} \right.\right. \\
&\quad \left.\left. - \sum_{j=(k-1)+1}^{m} \frac{\nu_{k-1}}{\nu_j} C_{jk-1} A_{k-1} \right] \right| \\
&\leq \max_{z\in[0,T]} \left| \sum_{k=q}^{n-1} L^{-1} \left[-\alpha_k P_{i,k} + \sum_{j=1}^{k-1} C_{kj} A_k - \sum_{j=k+1}^{m} \frac{\nu_k}{\nu_j} C_{jk} A_k \right] \right| \\
&\leq M_7 \max_{z\in[0,T]} \left| -L^{-1} \sum_{k=q}^{n-1} P_{i,k} + L^{-1} \sum_{k=q}^{n-1} A_k - L^{-1} \sum_{k=q}^{n-1} A_k \right| \\
&\leq M_7 \max_{z\in[0,T]} \left| -L^{-1} \sum_{k=q}^{n-1} P_{i,k} \right|
\end{aligned}
$$

$$\begin{aligned}
&\leq M_7 \max_{z\in[0,T]} \left| \sum_{k=q}^{n-1} P_{i,k} \right| \max_{z\in I} \int_0^z d\tilde{z} \\
&\leq M_7 T \max_{z\in[0,T]} |S_{n-1} - S_{q-1}| \\
&\leq \bar{\gamma} \|S_{n-1} - S_{q-1}\|.
\end{aligned}$$

If we consider $n = r+1$ and $q = r$ then, we have $\|S_{r+1} - S_r\| \leq \bar{\gamma}\|S_r - S_{r-1}\| \leq \bar{\gamma}^2\|S_{r-1} - S_{r-2}\| \leq \cdots \leq \bar{\gamma}^r\|S_1 - S_0\|$. From the triangle inequality, we obtain

$$\begin{aligned}
\|S_n - S_q\| &\leq \|S_{q+1} - S_q\| + \|S_{q+2} - S_{q+1}\| + \cdots + \|S_n - S_{n-1}\| \\
&\leq \left(\bar{\gamma}^q + \bar{\gamma}^{q+1} + \bar{\gamma}^{q+2} + \cdots + \bar{\gamma}^{n-1}\right) \|S_1 - S_0\| \\
&\leq \bar{\gamma}^q \left(1 + \bar{\gamma} + \bar{\gamma}^2 + \ldots + \bar{\gamma}^{n-q-1}\right) \|S_1 - S_0\| \\
&\leq \bar{\gamma}^q \left(\frac{1 - \bar{\gamma}^{n-q}}{1 - \bar{\gamma}}\right) \|P_1\|.
\end{aligned}$$

Since $0 < \bar{\gamma} < 1$ we have $1 - \bar{\gamma}^{n-q} < 1$, and consequently

$$\|S_n - S_q\| \leq \frac{\bar{\gamma}^q}{1 - \bar{\gamma}} \|P_1\|. \tag{16}$$

Since $\|P_1\| < \infty$ we have $\|S_n - S_q\| \to 0$ as $q \to \infty$. Hence, we conclude that $\{S_n\}$ is a Cauchy sequence in Y. So the series converges and the proof is complete.

To end this subsection, we estimate the maximum absolute error of the truncated series (12).

PROPOSITION 3. The maximum absolute truncation error of the series solution (12) of problem (2) and (11) is estimated by

$$\max_{z\in[0,T]} \left| P_k(z) - \sum_{i=0}^{q} P_{k,i}(z) \right| \leq \frac{\bar{\gamma}^q}{1 - \bar{\gamma}} \max_{z\in[0,T]} |P_{k,1}(z)|, \quad k = 1, \ldots, m.$$

PROOF. From the previous proposition, we have $\|S_n - S_q\| \leq \frac{\bar{\gamma}^q}{1-\bar{\gamma}} \max_{z\in[0,T]} |P_{k,1}(z)|$. For each $z \in [0,T]$, we have $S_n(z) \to P_k(z)$ with $n \to \infty$, so we have

$$\|P_k - S_q\| \leq \frac{\bar{\gamma}^q}{1 - \bar{\gamma}} \max_{z\in[0,T]} |P_{k,1}(z)|,$$

and the maximum absolute truncation error in the interval I is estimated to be

$$\max_{z\in[0,T]} \left| P_k(z) - \sum_{i=0}^{q} P_{k,i}(z) \right| \leq \max_{z\in[0,T]} \frac{\bar{\gamma}^q}{1 - \bar{\gamma}} |P_{k,1}(z)|, \tag{17}$$

for $k = 1, \ldots, m$. This completes the proof.

5 The Raman System with One Pump and Two Signals

In this section, we compute a numerical example to see the rate of convergence of the proposed method. We present a table with the error between the two consecutive iteration of the solution at given instants.

For the special case of an amplifier composed by a forward single pump and two propagating signals, the solution is expressed as follows:

$$\begin{cases} P_1(z) = P_1(0) + L^{-1}\left[-\alpha_1 P_1 - \frac{\nu_1}{\nu_2} C_{21} N(P_1,P_2) - \frac{\nu_1}{\nu_3} C_{31} N(P_1,P_3)\right], \\ P_2(z) = P_2(0) + L^{-1}\left[-\alpha_2 P_2 + C_{21} N(P_1,P_2) - \frac{\nu_2}{\nu_3} C_{32} K(P_2,P_3)\right], \\ P_3(z) = P_3(0) + L^{-1}\left[-\alpha_3 P_3 + C_{31} N(P_1,P_3) + C_{32} N(P_2,P_3)\right]. \end{cases} \tag{18}$$

The Adomian decomposition method assumes a series solution of the unknown functions $P_1(z), P_2(z)$ and $P_3(z)$ are given by

$$P_1(z) = \sum_{n=0}^{\infty} P_{1,n}(z), \quad P_2(z) = \sum_{n=0}^{\infty} P_{2,n}(z), \quad P_3(z) = \sum_{n=0}^{\infty} P_{3,n}(z). \tag{19}$$

Substituting (19) with their initial conditions into (18) yields

$$\begin{aligned} \sum_{n=0}^{\infty} P_{1,n}(z) &= P_1(0) + L^{-1}\left[-\alpha_1 \sum_{n=0}^{\infty} P_{1,n} - \frac{\nu_1}{\nu_2} C_{21} N(P_1,P_2) - \frac{\nu_1}{\nu_3} C_{31} N(P_1,P_3)\right], \\ \sum_{n=0}^{\infty} P_{2,n}(z) &= P_2(0) + L^{-1}\left[-\alpha_2 \sum_{n=0}^{\infty} P_{2,n} + C_{21} N(P_1,P_2) - \frac{\nu_2}{\nu_3} C_{32} N(P_2,P_3)\right], \\ \sum_{n=0}^{\infty} P_{3,n}(z) &= P_3(0) + L^{-1}\left[-\alpha_3 \sum_{n=0}^{\infty} P_{3,n} + C_{31} N(P_1,P_3) + C_{32} N(P_2,P_3)\right], \end{aligned}$$

where the functions $N(P_1,P_2), N(P_1,P_3)$ and $N(P_2,P_3)$ are related to the nonlinear terms, and making use of (13) and (14), the nonlinear terms can be expressed in terms of Adomian polynomials as follows:

$$N(P_1,P_2) = \sum_{n=0}^{\infty} D_n(P_1,P_2),\ N(P_1,P_3) = \sum_{n=0}^{\infty} B_n(P_1,P_3),\ N(P_2,P_3) = \sum_{n=0}^{\infty} E_n(P_2,P_3),$$

where

$$D_n = \sum_{k=0}^{n} P_{1,k} P_{2,n-k}, \quad B_n = \sum_{k=0}^{n} P_{1,k} P_{3,n-k}, \quad \text{and} \quad E_n = \sum_{k=0}^{n} P_{2,k} P_{3,n-k}.$$

Identifying the zeroth components of $P_{1,0}, P_{2,0}$ and $P_{3,0}$, the remaining components $P_{1,n+1}, P_{2,n+1}$ and $P_{3,n+1}$, $n \geq 0$ are obtained recursively by

$$P_{1,n+1} = L^{-1}\left[-\alpha_1 P_{1,n} - \frac{\nu_1}{\nu_2} C_{21} D_n - \frac{\nu_1}{\nu_3} C_{31} B_n\right],$$

S_n^1	z=10	z=30	z=60	z=100
$n=9$	7.24065412 $\times 10^{-5}$	3.62767771 $\times 10^{-5}$	1.57307374$\times 10^{-5}$	4.749965358 $\times 10^{-6}$
$n=29$	5.98448706$\times 10^{-5}$	3.21863535 $\times 10^{-5}$	1.47838597$\times 10^{-5}$	4.650101884 $\times 10^{-6}$
$n=49$	4.88886873 $\times 10^{-5}$	2.84241024 $\times 10^{-5}$	1.38763283$\times 10^{-5}$	4.551655496$\times 10^{-6}$
S_n^2	z=10	z=30	z=60	z=100
$n=9$	3.37306181 $\times 10^{-5}$	1.69441802 $\times 10^{-5}$	7.36352147 $\times 10^{-6}$	2.22692930 $\times 10^{-6}$
$n=29$	2.79583822$\times 10^{-5}$	1.50615797 $\times 10^{-5}$	6.92718129$\times 10^{-6}$	2.18087075 $\times 10^{-6}$
$n=49$	2.29127375$\times 10^{-5}$	1.33276366 $\times 10^{-5}$	6.50867685$\times 10^{-6}$	2.13545488$\times 10^{-6}$
S_n^3	z=10	z=30	z=60	z=100
$n=9$	3.53614661 $\times 10^{-5}$	1.76694509 $\times 10^{-5}$	7.64510894 $\times 10^{-6}$	2.30479348 $\times 10^{-6}$
$n=29$	2.91478046$\times 10^{-5}$	1.56493360 $\times 10^{-5}$	7.17806355$\times 10^{-6}$	2.25557842 $\times 10^{-6}$
$n=49$	2.37393882 $\times 10^{-5}$	1.36936673$\times 10^{-5}$	6.73072054$\times 10^{-6}$	2.20707260 $\times 10^{-6}$

Table 1: l_1-error of S_n^1, S_n^2 and S_n^3 for $n \in \{9, 29, 49\}$.

$$\begin{aligned} P_{2,n+1} &= L^{-1}\left[-\alpha_2 P_{2,n} + C_{21}D_n - \frac{\nu_2}{\nu_3}C_{32}E_n\right], \\ P_{3,n+1} &= L^{-1}\left[-\alpha_3 P_{3,n} + C_{31}B_n + C_{32}E_n\right]. \end{aligned}$$

Substituting the above expressions in (19), it gives the exact solution in the closed form.

In Table 1, we present the l_1-norm of the difference of

$$S_n^1 = |P_{1,n+1} - P_{1,n}|, \quad S_n^2 = |P_{2,n+1} - P_{2,n}|, \quad S_n^3 = |P_{3,n+1} - P_{3,n}| \text{ for } n \in \{9, 29, 49\},$$

with the following data

$P_{1,0} = 0.8,\ P_{2,0} = 1 \times 10^{-3},\ P_{3,0} = P_{2,0};$

$\nu_1 = 2.053373 \times 10^{14} THz,\ \nu_2 = 1.960 \times 10^{14} THz,\ \nu_3 = 1.958 \times 10^{14} THz;$

$C_{12} = 0.2648178032 \times 10^{-3}, C_{13} = 0.2736146032 \times 10^{-3}, C_{23} = 0.002859233333 \times 10^{-3},$

$\alpha 1 = 5.419400495579674 \times 10^{-5}$ (a pump),

$\alpha 2 = 4.947848510246372 \times 10^{-5}$ (a signal), $\alpha 3 = 4.932588910154262 \times 10^{-5}$ (a signal),

which can be found in [8].

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