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Common Fixed Points Of Three Self-Mappings In Cone Metric Spaces*

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Abstract

Some results on the common fixed points of three self-mappings in cone metric spaces in which the cone is not necessarily normal are proved. Consequently, the results are generalizations, extensions and improvements of some previous results in the literature.

1 Introduction and Preliminaries

The fact that ordered Banach spaces, normal cones and topical functions have applications in optimization theory is one of the motivation for research in ordered linear metric spaces (see e.g. [10, 11]). In 2007, Huang and Zhang [7] introduced the concept of cone metric spaces, which is a generalization of that of metric spaces, by replacing the real numbers with ordered Banach spaces.

Huang and Zhang [7] proved some fixed point theorems for some contractive maps in normal cone metric spaces. The results have been generalized by some authors [12, 13, 15]. Vetro [15] and Abbas and Jungck [1] studied common fixed points for non-commuting mappings in normal cone metric spaces. However, there exists non-normal cone metric spaces [8]. Hence the recent results of Jungck et al. [8] on the common fixed points for weakly compatible pairs on cone metric spaces are generalizations as well as extensions of the result of Vetro [15] in that normality assumption is removed and the pair of maps considered are more general. Recently, Arshad et al. [3] proved a result on common fixed point for three self mappings satisfying generalized contractive type conditions in a non-normal cone metric space. The result is a generalization of the results in [1, 7, 13] involving single-valued maps and pairs of contractive maps in cone metric spaces. The result in this paper is an extension of the results in [3] in that we consider three self-mappings satisfying more general contractive conditions, and a generalization of the results of [8].

The following definitions are in the literature (e.g. see [7]).

Let E be a real Banach space and P a subset of E. P is called a cone if and only if (i) P is closed, nonempty, and $P \neq \{0\}$; (ii) $a, b \in R$, $a, b \geq 0$, $x, y \in P \Longrightarrow\ ax + by \in P$; and (iii) $P \bigcap (-P) = \{0\}$.

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For a given cone $P \subseteq E$, we can define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. $x < y$ will stand for $x \leq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.

The cone P is called normal if there is $M > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\|x\| \leq M\|y\|$.

The least positive number M satisfying the above is called the normal constant of P.

DEFINITION 1.1. Let X be a non-empty set. Suppose that $d : X \times X \to E$ satisfies
(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.
Then d is called a cone metric on X, and (X, d) is called a cone metric space.

EXAMPLE 1.2 [7]. Let $E = R^2$, $P = \{(x, y) \in E : x, y \geq 0\}$, $X = R$ and $d : X \times X \to E$ defined by $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Clearly, this example shows that cone metric spaces generalize metric spaces. We now give another example where E is a linear metric space that is not normable.

EXAMPLE 1.3 [12]. Let $E = \ell^p$, $(0 < p < 1)$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0$, for all $n\}$, (X, ρ) a metric space and $d : X \times X \to E$ defined by $d(x, y) = \{\frac{\rho(x,y)}{2^n}\}_{n\geq 1}$. Then (X, d) is a cone metric space.

DEFINITION 1.4. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent to $x \in X$, i.e. $\lim_{n\to\infty} x_n = x$.

DEFINITION 1.5. Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X. If for every $c \in E$ with $0 \ll c$ there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$, then $\{x_n\}$ is called a Cauchy sequence in X.

It is shown in [7] that a convergent sequence in a cone metric space (X, d) is a Cauchy sequence.

DEFINITION 1.6 [8]. A point $x \in X$ is called a coincident point of a family of self maps $\{f_i, i \in I\}$, if there exists a point w (called a point of coincidence) in X such that $w = f_i(x)$ for all $i \in I$. Self-maps f and g are said to be weakly compatible if they commute at their coincidence point, that is, if $fx = gx$ for some $x \in X$, then $fgx = gfx$.

REMARK 1.7. The concept of weak compatibility is known to be the most general among all commutativity concepts in fixed point theory. For example every pair of weakly commuting self-maps and each pair of compatible self-maps are weakly compatible, but the reverse is not always true. In fact, the notion of weakly compatible maps is more general than compatibility of type (A), compatibility of type (B), compatibility of type (C) and compatibility of type (P). For a review of those notions of commutativity, see [4].

PROPOSITION 1.8 [3]. Let X be a non empty set and the mappings $f, g, h : X \to X$ have a unique point of coincidence in X. If (f, h) and (g, h) are weakly compatible self-maps of X, then f, g, h have a unique common fixed point.

2 Main Results

We adopt the technique used in [3].

Let (X, d) be a cone metric space and f, g and h be self mappings on X such that $f(X) \cup g(X) \subseteq h(X)$. Suppose $x_o \in X$ and $x_1 \in X$ is chosen such that $hx_1 = fx_0$ and $x_2 \in X$ is chosen such that $hx_2 = gx_1$. Continuing in this way, the sequence $\{hx_n\}$ such that

$$\begin{aligned} hx_{2k+1} &= fx_{2k}, \\ hx_{2k+2} &= gx_{2k+1}, \quad k = 0, 1, 2, ... \end{aligned} \tag{1}$$

is called a $(f - g)$-sequence with initial point x_o.

The following result is an extension of [3, Proposition 3.2].

PROPOSITION 2.1. Let (X, d) be a cone metric space and $f, g, h : X \to X$ be mappings such that $f(X) \cup g(X) \subseteq h(X)$. Assume that

$$d(fx, gy) \leq a_1 d(hx, fx) + a_2 d(hy, gy) + a_3 d(hy, fx) + a_4 d(hx, gy) + a_5 d(hx, hy) \tag{2}$$

for all $x, y \in X$, $x \neq y$ where $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and that $d(fx, gx) < d(hx, fx) + d(hx, gx)$ for all $x \in X$, whenever $fx \neq gx$.
Then every $(f - g)$-sequence with initial point x_0 is a Cauchy sequence.

PROOF. Observe that if f and g satisfy (2), it also satisfies

$$d(fx, gy) \leq kd(hx, fx) + kd(hy, gy) + ld(hy, fx) + ld(hx, gy) + md(hx, hy) \tag{3}$$

for all $x, y \in X$ where $k, l, m \in [0, 1)$ and $2k+2l+m < 1$, $(2k = a_1+a_2,\ l = a_3+a_4,\ a_5 = m)$. Suppose $\{hx_n\}$ is a $(f - g)$-sequence with initial point x_0. Assume $hx_n \neq hx_{n+1}$ for all $n \in N$, then $x_n \neq x_{n+1}$ for all n. Using (3), we have

$$\begin{aligned} d(hx_{2k+1}, hx_{2k+2}) &= d(fx_{2k}, gx_{2k+1}) \\ &\leq kd(hx_{2k}, fx_{2k}) + kd(hx_{2k+1}, gx_{2k+1}) + ld(hx_{2k+1}, fx_{2k}) \\ &\quad +ld(hx_{2k}, gx_{2k+1}) + md(hx_{2k}, hx_{2k+1}) \\ &\leq kd(hx_{2k}, hx_{2k+1}) + kd(hx_{2k+1}, hx_{2k+2}) \\ &\quad +ld(hx_{2k}, hx_{2k+2}) + md(hx_{2k}, h_{2k+1}) \\ &\leq kd(hx_{2k}, hx_{2k+1}) + kd(hx_{2k+1}, hx_{2k+2}) + ld(hx_{2k}, hx_{2k+1}) \\ &\quad +ld(hx_{2k+1}, hx_{2k+2}) + md(hx_{2k}, h_{2k+1}) \\ &\leq (k + l + m)d(hx_{2k}, hx_{2k+1}) + (k + l)d(hx_{2k+1}, hx_{2k+2}). \end{aligned}$$

Hence,

$$(1 - k - l)d(hx_{2k+1}, hx_{2k+2}) \leq (k + l + m)d(hx_{2k}, hx_{2k+1}),$$

so that

$$d(hx_{2k+1}, hx_{2k+2}) \leq (\frac{k + l + m}{1 - k - l})^{2k+1} d(hx_0, hx_1). \tag{4}$$

Similarly,

$$d(hx_{2k+2}, hx_{2k+3}) \leq (\frac{k + l + m}{1 - k - l})d(hx_{2k+1}, hx_{2k+2}),$$

and hence

$$d(hx_{2k+2}, hx_{2k+3}) \leq (\frac{k+l+m}{1-k-l})^{2k+2} d(hx_0, hx_1). \tag{5}$$

Let $\lambda = (\frac{k+l+m}{1-k-l})$. Then $\lambda < 1$. Following the same argument in (3.9)-(3.17) of [3], if follows immediately that $\{hx_n\}$ is a Cauchy sequence.

REMARK 2.2. If $a_3 = a_4 = 0$ in Proposition 2.1, then we have Proposition 3.2 of [3].

THEOREM 2.3. Let (X, d) be a cone metric space and P an order cone and $f, g, h : X \to X$ be mappings such that $f(X) \cup g(X) \subseteq h(X)$. Assume the following conditions hold:

$$d(fx, gy) \leq a_1 d(hx, fx) + a_2 d(hy, gy) + a_3 d(hy, fx) + a_4 d(hx, gy) + a_5 d(hx, hy) \tag{6}$$

for all $x, y \in X$, $x \neq y$ where $a_1, a_2, a_3, a_4, a_5 \in [0, 1)$ and $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, and that $d(fx, gx) < d(hx, fx) + d(hx, gx)$ for all $x \in X$, whenever $fx \neq gx$. If $f(X)$ or $f(X) \cup g(X)$ is a complete subspace of X, then f, g and h have a unique point of coincidence. Furthermore, if (f, h) and (g, h) are both weakly compatible, then f, g and h have a unique common fixed point.

PROOF. Since f, g satisfy (6), they also satisfy (3), we will use (3). If $f(X)$ is a complete subspace of X, since by Proposition 2.1, a $(f-g)$-sequence $\{hx_n\}$, with the initial point x_0 is a Cauchy sequence, then there exist $u, v \in X$ such that $hx_n \to v = hu$. The same argument holds if $f(X) \cup g(X)$ is a complete subspace of X with $v \in f(X) \cup g(X)$. From

$$\begin{aligned}
d(hu, fu) &\leq d(hu, hx_{2n}) + d(hx_{2n}, fu) \\
&\leq d(hu, hx_{2n}) + d(fu, gx_{2n-1}) \\
&\leq d(v, hx_{2n}) + kd(hu, fu) + kd(hx_{2n-1}, gx_{2n-1}) \\
&\quad + ld(hx_{2n-1}, fu) + ld(hu, gx_{2n-1}) + md(hu, hx_{2n-1}) \\
&\leq d(v, hx_{2n}) + kd(hu, fu) + kd(hx_{2n-1}, hx_{2n}) \\
&\quad + ld(hx_{2n-1}, fu) + ld(v, hx_{2n}) + md(v, hx_{2n-1}) \\
&\leq d(v, hx_{2n}) + kd(hu, fu) + kd(hx_{2n-1}, v) \\
&\quad + kd(v, hx_{2n}) + ld(hx_{2n-1}, v) + ld(hu, fu) \\
&\quad + ld(v, hx_{2n}) + md(v, hx_{2n-1}),
\end{aligned}$$

we obtain

$$\begin{aligned}
d(hu, fu) &\leq \frac{1}{1-k-l}[d(v, hx_{2n}) + kd(hx_{2n-1}, v) + kd(v, hx_{2n}) \\
&\quad + ld(hx_{2n-1}, v) + ld(v, hx_{2n}) + md(v, hx_{2n-1})] \\
&= \frac{1+k+l}{1-k-l} d(v, hx_{2n}) + \frac{k+l+m}{1-k-l} d(v, hx_{2n-1}).
\end{aligned} \tag{7}$$

Suppose $0 \ll c$ and there exists $n_0 \in N$ such that $d(v, hx_{2n}) \ll \frac{c(1-k-l)}{2(1+k+l)}$ and $d(hx_{2n-1}, v) \ll \frac{c(1-k-l)}{2(k+l+m)}$ for all $n \geq n_0$. Therefore $d(hu, fu) \ll c$ from (7) and hence $d(hu, fu) \ll \frac{c}{r}$

for every $r \in N$. Since $\frac{c}{r} - d(hu, fu) \in intP$, and P is closed, then as $r \to \infty$, we have that $-d(hu, fu) \in P$. Since $d(hu, fu) > 0$, therefore $d(hu, fu) \in P$ and so $d(hu, fu) \in P \cap (-P) = \{0\}$. Hence $d(hu, fu) = 0$. Therefore $hu = fu$. Similarly, by using the inequality

$$d(hu, gu) \leq d(hu, hx_{2n+1}) + d(hx_{2n+1}, gu), \tag{8}$$

we can show that $hu = gu$. Thus $v = hu = fu = gu$ and hence we conclude that v is a point of coincidence of h, f and g.

The next is to show that the point of coincidence is unique. Assume there is another point of coincidence v^* in X such that $v^* = hu^* = fu^* = gu^*$, for some $u^* \in X$. It is easy to check, using (3) that

$$\begin{aligned} d(v, v^*) &= d(fu, gu^*) \\ &\leq kd(hu, fu) + kd(hu^*, gu^*) + ld(hu^*, fu) + ld(hu, gu^*) \\ &\quad + md(hu, hu^*) \\ &\leq kd(v, v) + kd(v^*, v^*) + ld(v^*, v) + ld(v, v^*) + md(v, v^*) \\ &\leq (2l + m)d(v, v^*). \end{aligned}$$

Since $2l+m < 1$, then $v = v^*$. Since (f, h), (g, h) are weakly compatible by assumption, v is the unique point of coincidence of h, f and g, then by Proposition 1.8, v is the unique common fixed point of h, f and g.

REMARK 2.4.
(i) If $a_3 = a_4 = 0$ in Theorem 2.3, then we have the Theorem 3.3 of [3].
(ii) If $f = g$, Theorem 2.3 gives the main result in [13] and Theorem 2.8 of [8].
(iii) If we choose $f = g$ and $a_3 = a_4$ in Theorem 2.3, then we have a generalization of Theorem 3.1 of [12].
(iv) If $a_3 = a_4 = 0$ in Theorem 2.3, then we have a generalization of the Theorem 1 of [15].
(v) If $a_1 = a_2 = a_3 = a_4 = 0$, Theorem 2.3 is a generalization of Theorem 1 of [7], Theorem 2.1 of [1] and Theorem 2.3 of [3] .
(vi) If $a_3 = a_4 = a_5 = 0$, Theorem 2.3 generalizes Theorem 3 of [7], Theorem 2.3 of [1] and Theorem 2.6 of [13].
(vii) If $a_1 = a_2 = a_5 = 0$ and Theorem 2.3 generalizes Theorem 5 of [7], Theorem 2.5 of [1] and Theorem 2.7 of [13].
(viii) If $f = g = h$ Theorem 2.3 gives the result of Hardy and Rogers [6], which is a generalization of the results of Chatterjea [5] and Kannan [9] among others.

The following proposition is needed for the next result.

PROPOSITION 2.5. Let (X, d) be a cone metric space and $f, g, h : X \to X$ be mappings such that $f(X) \cup g(X) \subseteq h(X)$. Assume that $d(fx, gy) \leq \lambda u$ where

$$u \in \{d(hx, hy), d(hx, fx), d(hy, gy), \frac{d(hy, fx) + d(hx, gy)}{2}\} \tag{9}$$

for all $x, y \in X$, $x \neq y$, $0 < \lambda < 1$, and that $d(fx, gx) < d(hx, fx) + d(hx, gx)$, for all $x \in X$, whenever $fx \neq gx$. Then every $(f - g)$-sequence with initial point x_0 is a Cauchy sequence.

PROOF. Suppose $\{hx_n\}$ is a $(f-g)$-sequence with initial point x_0. Assume $hx_n \neq hx_{n+1}$ for all $n \in N$, then $x_n \neq x_{n+1}$ for all n. Using (9), we have

$$\begin{aligned} d(hx_{2k+1}, hx_{2k+2}) &= d(fx_{2k}, gx_{2k+1}) \\ &\leq \lambda\{d(hx_{2k}, hx_{2k+1}), d(hx_{2k}, fx_{2k}), d(hx_{2k+1}, gx_{2k+1}), \\ &\quad \frac{d(hx_{2k+1}, fx_{2k}) + d(hx_{2k}, gx_{2k+1})}{2}\} \\ &\leq \lambda\{d(hx_{2k}, hx_{2k+1}), d(hx_{2k+1}, hx_{2k+2}), \frac{d(hx_{2k}, hx_{2k+2})}{2}\}. \end{aligned}$$

It suffices to look at the following cases:

Case 1: $d(hx_{2k+1}, hx_{2k+2}) \leq \lambda d(hx_{2k}, hx_{2k+1})$.

Case 2: $d(hx_{2k+1}, hx_{2k+2}) \leq \lambda \frac{d(hx_{2k}, hx_{2k+2})}{2} \leq \lambda \frac{d(hx_{2k}, hx_{2k+1}) + d(hx_{2k+1}, hx_{2x+2})}{2}$. Thus, $d(hx_{2k+1}, hx_{2k+2}) \leq \frac{\lambda}{2-\lambda} d(hx_{2k}, hx_{2k+1})$.

Combining the two cases we have $d(hx_{2k+1}, hx_{2k+2}) \leq \lambda d(hx_{2k}, hx_{2k+1})$. Thus, $d(hx_{2k+1}, hx_{2k+2}) \leq \lambda^{2k+1} d(hx_o, hx_1)$. Similarly, $d(hx_{2k+2}, hx_{2k+3}) \leq \lambda^{2k+2} d(hx_o, hx_1)$. Following the same argument in (3.9)-(3.17) of [3], if follows immediately that $\{hx_n\}$ is a Cauchy sequence.

THEOREM 2.6. Let (X, d) be a cone metric space and P an order cone and $f, g, h : X \to X$ be mappings such that $f(X) \cup g(X) \subseteq h(X)$. Assume that $d(fx, gy) \leq \lambda u$ where

$$u \in \{d(hx, hy), d(hx, fx), d(hy, gy), \frac{d(hy, fx) + d(hx, gy)}{2}\} \tag{10}$$

for all $x, y \in X$, $x \neq y$, $0 < \lambda < 1$; and that $d(fx, gx) < d(hx, fx) + d(hx, gx)$ for all $x \in X$, whenever $fx \neq gx$. If $f(X)$ or $f(X) \cup g(X)$ is a complete subspace of X, then f, g and h have a unique point of coincidence. Furthermore, if (f, h) and (g, h) are both weakly compatible, then f, g and h have a unique common fixed point.

PROOF. If $f(X)$ is a complete subspace of X, since by Proposition 2.5, a $(f-g)$-sequence $\{hx_n\}$, with the initial point x_0 is a Cauchy sequence, then there exist $u, v \in X$ such that $hx_n \to v = hu$. The same argument holds if $f(X) \cup g(X)$ is a complete subspace of X with $v \in f(X) \cup g(X)$. From

$$\begin{aligned} d(hu, fu) &\leq d(hu, hx_{2n}) + d(hx_{2n}, fu) \\ &\leq d(hu, hx_{2n}) + d(fu, gx_{2n-1}) \\ &\leq d(v, hx_{2n}) + \lambda\{d(hu, hx_{2n-1}), d(hu, fu), d(hx_{2n-1}, gx_{2n-1}), \\ &\quad \frac{d(hx_{2n-1}, fu) + d(hu, gx_{2n-1})}{2}\} \\ &\leq d(v, hx_{2n}) + \lambda\{d(v, hx_{2n-1}), d(hu, fu), d(hx_{2n-1}, hx_{2n}), \\ &\quad \frac{d(hx_{2n-1}, fu) + d(v, hx_{2n})}{2}\} \\ &\leq d(v, hx_{2n}) + \lambda\{d(v, hx_{2n-1}), d(hu, fu), d(hx_{2n-1}, v) + d(v, hx_{2n}), \\ &\quad \frac{d(hx_{2n-1}, v) + d(hu, fu) + d(v, hx_{2n})}{2}\}, \end{aligned}$$

we consider the following cases:

Case 1: $d(hu,fu) \leq d(v,hx_{2n}) + \lambda d(v,hx_{2n-1})$,
Case 2: $d(hu,fu) \leq d(v,hx_{2n}) + \lambda d(hu,fu) \Rightarrow d(hu,fu) \leq \frac{1}{1-\lambda} d(v,hx_{2n})$,
Case 3: $d(hu,fu) \leq d(v,hx_{2n}) + \lambda\{d(hx_{2n-1},v) + d(v,hx_{2n})\}$, and thus $d(hu,fu) \leq (\lambda+1)d(v,hx_{2n}) + \lambda d(hx_{2n-1},v)$,
Case 4: $d(hu,fu) \leq d(v,hx_{2n}) + \frac{\lambda}{2}\{d(hx_{2n-1},v) + d(hu,fu) + d(v,hx_{2n})\}$, and thus $d(hu,fu) \leq \frac{2+\lambda}{2-\lambda} d(v,hx_{2n}) + \frac{\lambda}{2-\lambda} d(hx_{2n-1},v)$.
Combining the four cases, we have

$$d(hu,fu) \leq \frac{1}{\lambda+1} d(v,hx_{2n}) + \lambda d(v,hx_{2n-1}). \tag{11}$$

Suppose $0 \ll c$ and there exists $n_0 \in N$ such that $d(v,hx_{2n}) \ll \frac{c(\lambda+1)}{2}$ and $d(hx_{2n-1},v) \ll \frac{c}{2\lambda}$ for all $n \geq n_0$. Therefore $d(hu,fu) \ll c$ from (11). The fact that v is the common fixed point follows the same procedure as in the proof of Theorem 2.3. The uniqueness follows from the contractive definition (10).

REMARK 2.7. If $g = f$, we have Theorem 2.1 of [8] which is a generalization of several results in the references of [8].

Following the same procedure in the proof of Proposition 2.5, the proof of the following proposition follows easily.

PROPOSITION 2.8. Let (X,d) be a cone metric space and $f,g,h : X \to X$ be mappings such that $f(X) \cup g(X) \subseteq h(X)$. Assume that $d(fx,gy) \leq \lambda u$ where

$$u \in \{d(hx,hy), \frac{d(hx,fx)+d(hy,gy)}{2}, \frac{d(hy,fx)+d(hx,gy)}{2}\} \tag{12}$$

for all $x,y \in X$, $x \neq y$, $0 < \lambda < 1$; and that $d(fx,gx) < d(hx,fx) + d(hx,gx)$ for all $x \in X$, whenever $fx \neq gx$. Then every $(f-g)$-sequence with initial point x_0 is a Cauchy sequence.

We now state the following Theorem.

THEOREM 2.9. Let (X,d) be a cone metric space and P an order cone and $f,g,h : X \to X$ be mappings such that $f(X) \cup g(X) \subseteq h(X)$. Assume that $d(fx,gy) \leq \lambda u$ where

$$u \in \{d(hx,hy), \frac{d(hx,fx)+d(hy,gy)}{2}, \frac{d(hy,fx)+d(hx,gy)}{2}\} \tag{13}$$

for all $x,y \in X$, $x \neq y$, $0 < \lambda < 1$; and that $d(fx,gx) < d(hx,fx) + d(hx,gx)$ for all $x \in X$, whenever $fx \neq gx$. If $f(X)$ or $f(X) \cup g(X)$ is a complete subspace of X, then f, g and h have a unique point of coincidence. Furthermore, if (f,h) and (g,h) are both weakly compatible, then f, g and h have a unique common fixed point.

PROOF. If $f(X)$ is a complete subspace of X, since by Proposition 2.8, a $(f-g)$-sequence $\{hx_n\}$, with the initial point x_0 is a Cauchy sequence, then there exist $u,v \in X$ such that $hx_n \to v = hu$. The same argument holds if $f(X) \cup g(X)$ is a complete subspace of X with $v \in f(X) \cup g(X)$. In view of Theorem 2.6, it is sufficient to consider the case $d(fx,gy) \leq \lambda \frac{d(hx,fx)+d(hy,gy)}{2}$. Then

$$d(hu,fu) \leq d(hu,hx_{2n}) + d(hx_{2n},fu)$$

$$\begin{aligned} &\leq d(hu, hx_{2n}) + d(fu, gx_{2n-1}) \\ &\leq d(v, hx_{2n}) + \frac{\lambda}{2}(d(hu, fu) + d(hx_{2n-1}, gx_{2n-1})) \\ &\leq d(v, hx_{2n}) + \frac{\lambda}{2}(d(hu, fu) + d(hx_{2n-1}, hx_{2n})). \end{aligned}$$

After computing, we have $d(hu, fu) \leq \frac{2+\lambda}{2-\lambda} d(v, hx_{2n}) + \frac{\lambda}{2-\lambda} d(hx_{2n-1}, v)$. Suppose $0 \ll c$ and there exists $n_0 \in N$ such that $d(v, hx_{2n}) \ll \frac{c(2-\lambda)}{2(2+\lambda)}$ and $d(hx_{2n-1}, v) \ll \frac{c(2-\lambda)}{\lambda}$ for all $n \geq n_0$. Therefore $d(hu, fu) \ll c$ from (13). The fact that v is the common fixed point follows the same procedure as in the proof of Theorem 2.3. The uniqueness follows from the contractive definition (9).

REMARK 2.10. If $g = f$, we have Theorem 2.2 of [8] which is a generalization of several results in the references of [8].

EXAMPLE 2.11. Let $X = R$, $E = \ell^p$, $(0 < p < \infty)$, $P = \{\{x_n\}_{n\geq 1} \in E : x_n \geq 0,$ for all $n\}$. Let $d : X \times X \to E$ be defined by $d(x, y) = \{\frac{|x-y|}{2^n}\}_{n\geq 1}$. Then (X, d) is a cone metric space. Consider the mapping f, g and h defined as:

$$fx = \begin{cases} \frac{1}{1+\alpha} + \beta & x \neq 0 \\ 0, & x = 0 \end{cases} \quad gx = \begin{cases} \frac{1}{1-\alpha} - \beta & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$h(x) = \alpha x \quad \text{for all } x$$

where $\alpha > 1$ and $\beta > 0$. It is easy to check that

$$d(fx, gy) \leq a d(hx, hy)$$

for all $x, y \in X$, where $a = \frac{1}{(1+\alpha)\alpha} \in (0, 1)$. The only point of coincidence of f, g and h is 0. Also, the pair mappings (f, h) and (g, h) commutes at the point of coincidence and therefore are weakly compatible. All the conditions of Theorem 2.3 are satisfied and therefore f, g and h have a unique common fixed point which is 0.

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