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Uniqueness And Weighted Value Sharing Of Meromorphic Functions*

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Abstract

We study the uniqueness of meromorphic functions concerning nonlinear differential polynomials with weighted value sharing method and prove a uniqueness theorem which improves and generalizes a recent result in [16].

1 Introduction

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [6], [14] and [15]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions. For $a \in \mathbb{C} \cup \{\infty\}$ we say that f and g share the value a CM (counting multiplicities) if $f - a$ and $g - a$ have the same zeros with the same multiplicities and we say that f and g share the value a IM (ignoring multiplicities) if we do not consider the multiplicities.

Throughout this paper, we need the following definition.

$$\Theta(a,f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, W.K. Hayman proved that if f is a transcendental meromorphic function and $n(\geq 3)$ is a positive integer, then $f^n f' = 1$ has infinitely many solutions (see [5, Corollary of Theorem 9]). Corresponding to which, the following result was obtained by Fang and Hua [3] and by Yang and Hua [13] respectively.

THEOREM A. Let $f(z)$ and $g(z)$ be two nonconstant entire functions, $n \geq 6$ be a positive integer. If $f^n f'$ and $g^n g'$ share 1 CM, then either $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$,

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where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = -1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^{n+1} = 1$.

Fang [4] considered about the k-th derivative instead of the first derivative and proved the following theorems.

THEOREM B. Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n > 2k + 4$. If $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.

THEOREM C. Let $f(z)$ and $g(z)$ be two nonconstant entire functions, and let n, k be two positive integers with $n \geq 2k + 8$. If $[f^n(f-1)]^{(k)}$ and $[g^n(g-1)]^{(k)}$ share 1 CM, then $f(z) \equiv g(z)$.

Recently Bhoosnurmath and Dyavanal [2] also considered the uniqueness of meromorphic functions corresponding to the k-th derivative of a linear polynomial expression. They proved the following theorem.

THEOREM D. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, k be two positive integers with $n > 3k + 8$. If $[f^n(z)]^{(k)}$ and $[g^n(z)]^{(k)}$ share 1 CM, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.

Naturally, one may ask the following question: Is it possible in any way to relax the nature of sharing the value 1 in the above results?

It is worth mentioning that in the above area some investigations has already been carried out by Zhang and Lu [16]. To state the result we require the following definition known as weighted sharing of values introduced by Lahiri [8, 9] which measure how close a shared value is to being shared CM or to being shared IM.

DEFINITION 1. Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f where an a-point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

The definition implies that if f, g share a value a with weight k, then z_0 is an a-point of f with multiplicity $m(\leq k)$ if and only if it is an a-point of g with multiplicity $m(\leq k)$ and z_0 is an a-point of f with multiplicity $m(> k)$ if and only if it is an a-point of g with multiplicity $n(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ and (a, ∞) respectively.

Zhang and Lu [16] proved the following theorem.

THEOREM E. Let $f(z)$ and $g(z)$ be two nonconstant transcendental meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$, $l(\geq 0)$ be three integers. Suppose that $[f^n]^{(k)}$ and $[g^n]^{(k)}$ share $(1, l)$, if $l \geq 2$ and $n > 3k + 8$ or if $l = 1$ and $n > 5k + 11$ or if $l = 0$ and $n > 9k + 14$, then either $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f(z) \equiv tg(z)$ for a constant t such that $t^n = 1$.

Regarding Theorem E, it is natural to ask the following questions.

QUESTION 1. What can be said about the relation between two nonconstant meromorphic functions f and g, if $\{f^n(f^m-a)\}^{(k)}$ and $\{g^n(g^m-a)\}^{(k)}$ share (b,l) for a nonzero constant b?

QUESTION 2. What can be said about the relation between two nonconstant meromorphic functions f and g, if $\{f^n(f-1)^m\}^{(k)}$ and $\{g^n(g-1)^m\}^{(k)}$ share (b,l) for a nonzero constant b?

Recently Liu [11] proved the following theorem relating with Question 1.

THEOREM F. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, and let n, m and k be three positive integers, and λ, μ be two constants such that $|\lambda|+|\mu|\neq 0$. If $[f^n(\mu f^m+\lambda)]^{(k)}$ and $[g^n(\mu g^m+\lambda)]^{(k)}$ share $(1,l)$, and one of the following conditions holds: (a) $l\geq 2$ and $n>3m^{**}+3k+8$; (b) $l=1$ and $n>4m^{**}+5k+10$; (c) $l=0$ and $n>6m^{**}+9k+14$. Then

(i) when $\lambda\mu\neq 0$, if $m\geq 2$ and $\delta(\infty,f)>\frac{3}{m+n}$, then $f(z)\equiv g(z)$; if $m=1$ and $\Theta(\infty,f)>\frac{3}{n+1}$, then $f(z)\equiv g(z)$; and

(ii) when $\lambda\mu=0$, if $f(z)\neq\infty$ and $g(z)\neq\infty$, then either $f(z)\equiv tg(z)$, where t is a constant satisfying $t^{n+m^{**}}=1$, or $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k\lambda^2(c_1c_2)^{n+m^{**}}[(n+m^{**})c]^{2k}=1$ or $(-1)^k\mu^2(c_1c_2)^{n+m^{**}}[(n+m^{**})c]^{2k}=1$, where $m^{**}=\chi_\mu m$,

$$\chi_\mu=\begin{cases}0 & \text{if } \mu=0\\ 1 & \text{if } \mu\neq 0.\end{cases}$$

In this paper, we will prove the following theorem which not only provide a supplementary result of Theorem D, also improve and generalize Theorem E. Moreover, our theorem deal with Question 2 also.

THEOREM 1. Let $f(z)$ and $g(z)$ be two transcendental meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ and $l(\geq 0)$ be four integers. Let $[f^n(f-1)^m]^{(k)}$ and $[g^n(g-1)^m]^{(k)}$ share (b,l) for a nonzero constant b. Then

(i) when $m=0$, if $f(z)\neq\infty$, $g(z)\neq\infty$ and $l\geq 2$, $n>3k+8$ or $l=1$, $n>5k+10$ or $l=0$, $n>9k+14$, then either $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=b^2$ or $f(z)\equiv tg(z)$ for a constant t such that $t^n=1$;

(ii) when $m=1$ and $\Theta(\infty,f)>\frac{2}{n}$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv b^2$ or $f(z)\equiv g(z)$ provided one of $l\geq 2$, $n>3k+11$ or $l=1$, $n>5k+14$ or $l=0$, $n>9k+20$ holds; and

(iii) when $m\geq 2$ and $l\geq 2$, $n>3k+m+10$ or $l=1$, $n>5k+2m+12$ or $l=0$, $n>9k+4m+16$, then either $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv b^2$ or $f(z)\equiv g(z)$ or $f(z)$ and $g(z)$ satisfy the algebraic equation $R(f,g)=0$, where

$$R(x,y)=x^n(x-1)^m-y^n(y-1)^m.$$

REMARK 1. The possibility $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv b^2$ of Theorem 1 does not arise for $k=1$.

REMARK 2. Obviously Theorem 1 is an improvement of Theorem E for $m=0$ and $l=1$.

Though the standard definitions and notations of the value distribution theory are available in [6], we explain some definitions and notations which are used in the paper.

DEFINITION 2. [7] Let p be a positive integer and $b \in \mathbb{C} \cup \{\infty\}$. Then by $N(r, b; f \mid\leq p)$ we denote the counting function of those b-points of f (counted with multiplicities) whose multiplicities are not greater than p. By $\overline{N}(r, b; f \mid\leq p)$ we denote the corresponding reduced counting function.

In an analogous manner we define $N(r, b; f \mid\geq p)$ and $\overline{N}(r, b; f \mid\geq p)$.

DEFINITION 3. [10] Let k be a positive integer or infinity. We denote by $N_k(r, b; f)$ the counting function of b-points of f, where an b-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. That is

$$N_k(r, b; f) = \overline{N}(r, b; f) + \overline{N}(r, b; f \mid\geq 2) + ... + \overline{N}(r, b; f \mid\geq k).$$

DEFINITION 4. For $b \in \mathbb{C} \cup \{\infty\}$ we put

$$\delta_k(b, f) = 1 - \limsup_{r\to\infty} \frac{N_k(r, b; f)}{T(r, f)}.$$

2 Lemmas

In this section we present some lemmas which will be needed to prove the theorems.

LEMMA 1. [12] Let $f(z)$ be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + ... + a_n f^n$, where a_0, a_1, a_2, ... ,a_n are constants and $a_n \neq 0$. Then

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

LEMMA 2. [11] Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions, $k(\geq 1)$, $l(\geq 0)$ be integers. Suppose that $f^{(k)}$ and $g^{(k)}$ share $(1, l)$. If one of the following conditions holds, then either $f^{(k)}(z)g^{(k)}(z) \equiv 1$ or $f(z) \equiv g(z)$.

(i) $l \geq 2$ and $\Delta_1 = 2\Theta(\infty, f) + (k+2)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) > k + 7$;

(ii) $l = 1$ and $\Delta_2 = (k+3)\Theta(\infty, f) + (k+2)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + \delta_{k+1}(0, g) > 2k + 9$;

(iii) $l = 0$ and $\Delta_3 = (2k+4)\Theta(\infty, f) + (2k+3)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 3\delta_{k+1}(0, f) + 2\delta_{k+1}(0, g) > 4k + 13$.

LEMMA 3. Let f and g be two nonconstant meromorphic functions and $n(\geq 1)$, $m(\geq 1)$, $k(\geq 1)$ be three integers. Then

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv b^2,$$

for $k = 1$ and $n \geq m + 3$.

PROOF. If possible, let

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv b^2,$$

for $k=1$. That is,

$$f^{n-1}(f-1)^{m-1}(cf-d)f'g^{n-1}(g-1)^{m-1}(cg-d)g'\equiv b^2,$$

where $c=n+m$ and $d=n$. Let z_0 be a 1-point of f with multiplicity $p(\geq 1)$, and a pole of g with multiplicity $q(\geq 1)$ such that

$$mp-1=(n+m)q+1,$$

i.e.,

$$mp=(n+m)q+2\geq n+m+2,$$

i.e.,

$$p\geq \frac{n+m+2}{m}.$$

Let z_1 be a zero of $cf-d$ with multiplicity $p_1(\geq 1)$, and a pole of g with multiplicity $q_1(\geq 1)$ such that

$$2p_1-1=(n+m)q_1+1,$$

i.e.,

$$p_1\geq \frac{n+m+2}{2}.$$

Let z_2 be a zero of f with multiplicity $p_2(\geq 1)$, and a pole of g with multiplicity $q_2(\geq 1)$. Then

$$np_2-1=(n+m)q_2+1. \tag{1}$$

From (1) we get

$$mq_2+2=n(p_2-q_2)\geq n,$$

i.e.,

$$q_2\geq \frac{n-2}{m}.$$

Thus from (1) we get

$$np_2=(n+m)q_2+2\geq \frac{(n+m)(n-2)}{m}+2,$$

i.e.,

$$p_2\geq \frac{n+m-2}{m}.$$

Since a pole of f is either a zero of $g(g-1)(cg-d)$ or a zero of g', we have

$$\begin{aligned}\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g)+\overline{N}(r,1;g)+\overline{N}\left(r,\frac{d}{c};g\right)+\overline{N}_0(r,0;g')+S(r,f)+S(r,g)\\ &\leq \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)T(r,g)+\overline{N}_0(r,0;g')+S(r,f)+S(r,g),\end{aligned}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g-1)(cg-d)$.

Then by the second fundamental theorem of Nevanlinna we get

$$\begin{aligned}2T(r,f) &\leq \overline{N}(r,0;f)+\overline{N}(r,1;f)+\overline{N}\left(r,\frac{d}{c};f\right)+\overline{N}(r,\infty;f)-\overline{N}_0(r,0;f')+S(r,f)\\ &\leq \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}-\overline{N}_0(r,0;f')+\\ &\quad \overline{N}_0(r,0;g')+S(r,f)+S(r,g). \qquad (2)\end{aligned}$$

Similarly we get

$$\begin{aligned}2T(r,g) &\leq \left(\frac{m+2}{n+m+2}+\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;f')\\ &\quad -\overline{N}_0(r,0;g')+S(r,f)+S(r,g). \qquad (3)\end{aligned}$$

Adding (2) and (3) we obtain

$$\left(1-\frac{m+2}{n+m+2}-\frac{m}{n+m-2}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

which is a contradiction for $n\geq m+3$. This proves the lemma.

LEMMA 4. [1] Let f, g be two nonconstant meromorphic functions and let, $k\geq 1$ and $n>3k+8$ be two integers. If $[f^n]^{(k)}[g^n]^{(k)}\equiv b^2$, where $b(\neq 0)$, be a constant, then $f(z)=c_1e^{cz}$, $g(z)=c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k}=b^2$.

3 Proof of Theorem 1

We consider $F(z)=\frac{f^n(f-1)^m}{b}$ and $G(z)=\frac{g^n(g-1)^m}{b}$. Using Lemma 1, we get

$$\begin{aligned}\Theta(\infty,F) &= 1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,\infty;F)}{T(r,F)}=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}\left(r,\infty;\frac{f^n(f-1)^m}{b}\right)}{(m+n)T(r,f)}\\ &\geq 1-\limsup_{r\longrightarrow\infty}\frac{T(r,f)}{(m+n)T(r,f)}\geq\frac{n+m-1}{m+n}. \qquad (4)\end{aligned}$$

Similarly

$$\Theta(\infty,G)\geq\frac{n+m-1}{m+n}. \qquad (5)$$

$$\begin{aligned}\Theta(0,F) &= 1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}(r,0;F)}{T(r,F)}=1-\limsup_{r\longrightarrow\infty}\frac{\overline{N}\left(r,0;\frac{f^n(f-1)^m}{b}\right)}{(m+n)T(r,f)}\\ &\geq 1-\limsup_{r\longrightarrow\infty}\frac{(1+m^*)T(r,f)}{(m+n)T(r,f)}\geq\frac{n+m-1-m^*}{m+n}, \qquad (6)\end{aligned}$$

where

$$m^*=\begin{cases}0 & \text{if } m=0\\ 1 & \text{if } m\geq 1.\end{cases}$$

Similarly

$$\Theta(0,G) \geq \frac{n+m-1-m^*}{n+m}. \tag{7}$$

$$\begin{aligned}\delta_{k+1}(0,F) &= 1-\limsup_{r\longrightarrow\infty}\frac{N_{k+1}(r,0;F)}{T(r,F)} = 1-\limsup_{r\longrightarrow\infty}\frac{N_{k+1}\left(r,0;\frac{f^n(f-1)^m}{b}\right)}{(m+n)T(r,f)}\\ &\geq 1-\limsup_{r\longrightarrow\infty}\frac{(k+m+1)T(r,f)}{(m+n)T(r,f)} \geq \frac{n-k-1}{m+n}. \end{aligned}\tag{8}$$

Similarly

$$\delta_{k+1}(0,G) \geq \frac{n-k-1}{m+n}. \tag{9}$$

Since $F^{(k)}$ and $G^{(k)}$ share $(1,l)$, we discuss the following three cases:

Case 1. Let $l \geq 2$. From (4)-(9) we obtain

$$\begin{aligned}\Delta_1 &= (k+4)\frac{n+m-1}{m+n}+2\frac{n+m-1-m^*}{m+n}+2\frac{n-k-1}{m+n}\\ &= \frac{1}{m+n}\left[(k+4)(n+m-1)+2(n+m-1-m^*)+2(n-k-1)\right].\end{aligned}$$

It is easily verified that $\Delta_1 > k+7$ provided $n > 3k+m+2m^*+8$. Since

$$3k+m+2m^*+8 = \begin{cases} 3k+8 & \text{if } m=0\\ 3k+11 & \text{if } m=1\\ 3k+m+10 & \text{if } m\geq 2,\end{cases}$$

by (i) of Lemma 2 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Case 2. Let $l = 1$. Then from (4)-(9) we obtain

$$\begin{aligned}\Delta_2 &= (2k+5)\frac{n+m-1}{m+n}+2\frac{n+m-1-m^*}{m+n}+3\frac{n-k-1}{m+n}\\ &= \frac{1}{m+n}\left[(2k+5)(n+m-1)+2(n+m-1-m^*)+3(n-k-1)\right].\end{aligned}$$

It is easily verified that $\Delta_2 > 2k+9$ provided $n > 5k+2m+2m^*+10$. Since

$$5k+2m+2m^*+10 = \begin{cases} 5k+10 & \text{if } m=0\\ 5k+14 & \text{if } m=1\\ 5k+2m+12 & \text{if } m\geq 2,\end{cases}$$

by (ii) of Lemma 2 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

Case 3. Let $l = 0$. Then as Case 1 and Case 2, it is easy to verify that $\Delta_3 > 4k+13$ when $n > 9k+4m+2m^*+14$. Since

$$9k+4m+2m^*+14 = \begin{cases} 9k+14 & \text{if } m=0\\ 9k+20 & \text{if } m=1\\ 9k+4m+16 & \text{if } m\geq 2,\end{cases}$$

by (iii) of Lemma 2 we obtain either $F^{(k)}G^{(k)} \equiv 1$ or $F \equiv G$.

We suppose that $F^{(k)}G^{(k)} \equiv 1$. That is

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv b^2. \tag{10}$$

Let $m = 0$. Since $f(z) \neq \infty$ and $g(z) \neq \infty$, by (10) and Lemma 4 we obtain $f(z) = c_1e^{cz}$, $g(z) = c_2e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = b^2$. Again by Lemma 3,

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv b^2,$$

for $k = 1$ and $m \geq 1$. Next we suppose that

$$F \equiv G.$$

i.e.,

$$f^n(f-1)^m \equiv g^n(g-1)^m. \tag{11}$$

Now we consider the following three subcases.

Subcase (i). Let $m = 0$. Then from (11) we get $f \equiv tg$ for a constant t such that $t^n = 1$.

Subcase (ii). Let $m = 1$. Then from (11) we obtain

$$f^n(f-1) \equiv g^n(g-1). \tag{12}$$

Suppose $f \not\equiv g$. Let $h = \frac{f}{g}$ be a constant. Then from (12) it follows that $h \neq 1$, $h^n \neq 1$, $h^{n+1} \neq 1$ and $g = \frac{1-h^n}{1-h^{n+1}}=$ constant, a contradiction. So we suppose that h is not a constant. Since $f \not\equiv g$, we have $h \not\equiv 1$. From (12) we obtain $g = \frac{1-h^n}{1-h^{n+1}}$ and $f = \left(\frac{1-h^n}{1-h^{n+1}}\right) h$. Hence it follows that

$$T(r,f) = nT(r,h) + S(r,f).$$

Again by second fundamental theorem of Nevanlinna, we have

$$\overline{N}(r,\infty;f) = \sum_{j=1}^{n} \overline{N}(r,\alpha_j;h) \geq (n-2)T(r,h) + S(r,f),$$

where α_j $(\neq 1)$ $(j = 1,2,...,n)$ are distinct roots of the equation $h^{n+1} = 1$. So we obtain

$$\Theta(\infty,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,\infty;f)}{T(r,f)} \leq \frac{2}{n},$$

which contradicts the assumption $\Theta(\infty;f) > \frac{2}{n}$. Thus $f \equiv g$.

Subcase (iii). Let $m \geq 2$. Then from (11) we obtain

$$f^n[f^m + \cdots + (-1)^i\ {}^mC_{m-i}f^{m-i} + \cdots + (-1)^m] = g^n[g^m + \cdots + (-1)^i\ {}^mC_{m-i}g^{m-i} + \cdots + (-1)^m]. \tag{13}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ in (13) we obtain

$$g^{n+m}(h^{n+m}-1)+\cdots+(-1)^i \ {}^mC_{m-i}g^{n+m-i}(h^{n+m-i}-1)+\cdots+(-1)^m g^n(h^n-1)=0,$$

which imply $h = 1$. Hence $f \equiv g$.

If h is not a constant, then from (13) we can say that f and g satisfy the algebraic equation $R(f, g) = 0$, where

$$R(x, y) = x^n(x-1)^m - y^n(y-1)^m.$$

This completes the proof of the theorem.

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References

[1] A. Banerjee, Uniqueness of certain non-linear differential polynomials sharing the same value, Int. J. Pure Appl. Math., 48(1)(2008), 41–56.

[2] S. S. Bhoosnurmath and R. S. Dyavanal, Uniqueness and value sharing of meromorphic functions, Comput. Math. Appl., 53(2007), 1191–1205.

[3] M. L. Fang and X. H. Hua, Entire functions that share one value, J. Nanjing Univ. Math. Biquarterly, 13(1)(1996), 44–48.

[4] M. L. Fang, Uniqueness and value sharing of entire functions, Comput. Math. Appl., 44(2002), 828–831.

[5] W. K. Hayman, Picard values of meromorphic functions and their derivatives, Ann. of Math., 70(1959), 9–42.

[6] W. K. Hayman, Meromorphic Functions, The Clarendon Press, Oxford, 1964.

[7] I. Lahiri, Value distribution of certain differential polynomials, Int. J. Math. Math. Sci., 28(2)(2001), 83–91.

[8] I. Lahiri, Weighted sharing and uniqueness of meromorphic functions, Nagoya Math. J., 161(2001), 193–206.

[9] I. Lahiri, Weighted value sharing and uniqueness of meromorphic functions, Complex Var. Theory Appl., 46(2001), 241–253.

[10] I. Lahiri, Weighted sharing of three values, Z. Anal. Anwendungen, 23(2)(2004), 237–252.

[11] L. Liu, Uniqueness of meromorphic functions and differential polynomials, Comput. Math. Appl., 56 (2008), 3236–3245.

[12] C. C. Yang, On deficiencies of differential polynomials II, Math. Z., 125(1972), 107–112.

[13] C. C. Yang and X. H. Hua, Uniqueness and value sharing of meromorphic functions, Ann. Acad. Sci. Fenn. Math., 22(1997), 395–406.

[14] L. Yang, Value Distribution Theory, Springer-Verlag, Berlin, 1993.

[15] H. X. Yi and C. C. Yang, Uniqueness Theory of Meromorphic Functions, Science Press, Beijing, 1995.

[16] T. Zhang and W. Lu, Uniqueness theorems on meromorphic functions sharing one value, Comput. Math. Appl., 55(2008), 2981–2992.