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Existence Of Solutions And Controllability Of Nonlinear Mixed Integrodifferential Equation With Nonlocal Condition*

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Abstract

The present paper investigates the existence of mild solutions of a nonlinear mixed Volterra-Fredholm integrodifferential equation with nonlocal condition in Banach spaces. Further sufficient condition for the controllability of integrodifferential equation is established. The approach used is the Schauder fixed point theorem with the theory of resolvent operators.

1 Introduction

Let X be a Banach space with norm $\|\cdot\|$. Let $Z = C(J, X)$ be the Banach space of all continuous functions from J into X endowed with the supremum norm

$$\|x\|_Z = \sup\{\|x(t)\| : t \in J\}$$

and $B(X)$ denotes the Banach space of bounded linear operators from X into itself.

Motivated by the work of [20], in this paper we consider the following nonlinear mixed Volterra-Fredholm integrodifferential equation of the form:

$$x^{'}(t) = A(t)\Big(x(t) + \int_0^t Q(t,s)x(s)ds\Big) + f(t, x(t), \int_0^t k(t,s,x(s))ds, \int_0^b h(t,s,x(s))ds), \tag{1}$$

$$x(0) + g(x) = x_0, \tag{2}$$

where $t \in J = [0, b]$, the unknown $x(\cdot)$ takes values in the Banach space X, and x_0 is a given element of X. Here $A(t)$ is a closed linear operator on X with dense domain $D(A)$, which is independent of t. $Q(t,s), t, s \in J$, is a bounded operator in X. The nonlinear functions $f : J \times X \times X \times X \to X$, $g : Z \to X$, $k,\ h : J \times J \times X \to X$ are continuous functions. We define the following sets:

$$B_r = \{x \in X : \|x\| \le r\} \quad \text{and} \quad E_r = \{z \in Z : \|z\|_Z \le r\},$$

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where $r > 0$ is defined below.

The nonlocal condition, which is a generalization of the classical initial condition, was motivated by physical problems. The problem of existence of solutions of evolution equation with nonlocal conditions in Banach space was first studied by [5] and he investigated the existence and uniqueness of mild, strong and classical solutions of the nonlocal Cauchy problem. As indicated in [5, 10] and the references therein, the nonlocal condition $y(0) + g(y) = y_0$ can be applied in physics with better effect than the classical condition $y(0) = y_0$. For example, in [10], the author used

$$g(y) = \sum_{i=1}^{p} c_i y(t_i),$$

where$c_i, \quad i = 1, 2, ..., p$ and $0 < t_1 < t_2 < \cdots \leq b$, to describe the diffusion phenomenon of a small amount of gas in a transparent tube. In this case, the above explanation allows the additional measurements at $t_i, \quad i = 1, 2, ..., p$. The study of differential and integrodifferential equations in abstract spaces with nonlocal conditions have received much attention in recent years. We refer to the papers [6, 7, 16, 19].

The objective of the present paper is to generalize the results reported in [13, 14, 16, 17] and our approach to the conditions on functions are different. The papers reported in [3, 20] are also special cases of the problem (1)–(2) when the function $\sigma(t) = t$. We first investigate the existence of mild solutions of the problem (1)–(2). The main tool employed in our analysis is based on the Schauder fixed point theorem and the theory of resolvent operators. We also study the nonlocal controllability problem for the above equation.

The paper is organized as follows. In section 2, we present the preliminaries and the hypotheses. Section 3 deals with the main result. Section 4 concerns with the controllability of integrodifferential equation. In section 5, we give an example to illustrate the applications of our results.

2 Preliminaries and Main Results

Before proceeding to our results, we shall set forth some preliminaries and hypotheses that will be used in our subsequent discussion.

DEFINITION. A resolvent operator for (1)–(2) is a bounded operator-valued function $R(t, s) \in B(X)$, $0 \leq s \leq t \leq b$, having the following properties:

(a) $R(t, s)$ is strongly continuous in s and t, $R(s, s) = I$, the identity operator on X, $0 \leq s \leq b$, and $\|R(t, s)\| \leq Me^{\beta(t-s)}$ for some constants M and β.

(b) $R(t, s)Y \subset Y$, $R(t, s)$ is strongly continuous in s and t on Y, and Y is the Banach space formed from $D(A)$, the domain $A(t)$, endowed with the graph norm.

(c) For each $y \in Y$, $R(t, s)y$ is continuously differential in $s \in J$ and

$$\frac{\partial}{\partial s} R(t, s)y = -R(t, s)A(s)y - \int_s^t R(t, \tau)Q(\tau, s)A(s)y d\tau.$$

(*d*) For each $y \in Y$, and $s \in J$, $R(t,s)y$ is continuously differential in $t \in J$ and

$$\frac{\partial}{\partial t}R(t,s)y = R(t,s)A(t)y + \int_s^t R(t,\tau)Q(\tau,s)A(t)y d\tau$$

with $\frac{\partial}{\partial s}R(t,s)y$ and $\frac{\partial}{\partial t}R(t,s)y$ are strongly continuous on $0 \leq s \leq t \leq b$. Here, $R(t,s)$ can be deduced from the evolution operator of the generator $A(t)$.

DEFINITION. A continuous solution $x(\cdot): J \to X$ is said to be a mild solution of problem (1)–(2) on J if for $x_0 \in X$, it satisfies the following integral equation

$$\begin{aligned} x(t) = R(t,0)[x_0 - g(x)] + \int_0^t R(t,s) f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \\ \int_0^b h(s,\tau,x(\tau))d\tau)ds. \end{aligned} \tag{3}$$

We need the following theorem (known as Schauder fixed point theorem [18], p-37) for further discussion:

THEOREM 1. Let S be a bounded, closed and convex subset of a Banach space X. If $f \in \mathcal{C}(S,S)$-set of all compact maps from S into S, then f has at least one fixed point.

We list the following hypotheses for our convenience.

(H_1) The resolvent operator $R(t,s)$ is compact when $t - s > 0$ and there exists a positive constant M_1 such that

$$\|R(t,s)\| \leq M_1.$$

(H_2) There are constants L_1, K_1 and H_1 such that

$$L_1 = \max_{t \in J} \|f(t,0,0,0)\|, \quad K_1 = \max_{0 \leq s \leq t \leq b} \|k(t,s,0)\|, \quad H_1 = \max_{0 \leq s,t \leq b} \|h(t,s,0)\|.$$

(H_3) There exists a constant $G_1 > 0$ such that

$$\|g(x)\| \leq G_1, \quad \text{for} \quad x \in E_r, \quad g(\lambda x_1 + (1-\lambda)x_2) = \lambda g(x_1) + (1-\lambda)g(x_2),$$

for $x_i \in E_r$, $(i = 1,2)$, $\lambda \in (0,1)$.

(H_4) The set

$$\{x(0) : x \in E_r, x(0) + g(x) = x_0\}$$

where

$$M_1[\|x_0\| + G_1 + Lrb + LKrb^2 + LK_1b^2 + LHrb^2 + LH_1b^2 + L_1b] \leq r,$$

with $[M_1Lb + M_1LKb^2 + M_1LHb^2] < 1$, is precompact in X.

3 Existence Result

Now we shall prove the following result of existence of mild solution.

THEOREM 2. Assume that

(*i*) hypotheses (H_1)–(H_4) hold,

(*ii*) $f \in C(J \times X \times X \times X; X)$ and there exists a constant $L > 0$ such that

$$\|f(t, x_1, y_1, z_1) - f(t, x_2, y_2, z_2)\| \leq L(\|x_1 - x_2\| + \|y_1 - y_2\| + \|z_1 - z_2\|),$$

for $x_i, y_i, z_i \in B_r$, $i = 1, 2$ and $t \in J$.

(*iii*) $k,\ h \in C(J \times J \times X; X)$ and there exist constants $K,\ \ H > 0$ such that

$$\|k(t, s, x_1) - k(t, s, x_2)\| \leq K\|x_1 - x_2\|$$

and

$$\|h(t, s, x_1) - h(t, s, x_2)\| \leq H\|x_1 - x_2\|$$

for $x_i, y_i \in B_r$, $i = 1, 2$ and $t, s \in J$. Then problem (1)–(2) has a mild solution on J.

PROOF. We define the set E by $E = \{x \in Z : x \in E_r, \quad x(0) + g(x)) = x_0\}$. It is easy to see that E is a bounded closed convex subset of Z. Define a mapping $F : E \to E$ by

$$(Fx)(t) = R(t, 0)[x_0 - g(x)] + \int_0^t R(t, s) f(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau,$$
$$\int_0^b h(s, \tau, x(\tau)) d\tau) ds, \quad t \in J. \tag{4}$$

Since all the functions involved in the definition of the operator are continuous, the operator F is continuous. For $x \in E, t \in J$ and using hypotheses (H_1)–(H_4) and assumptions $(ii), (iii)$, we have

$$\begin{aligned}
\|(Fx)(t)\| \leq\ & M_1(\|x_0\| + G_1) + M_1 \int_0^t [\|f(s, x(s), \int_0^s k(s, \tau, x(\tau)) d\tau, \\
& \int_0^b h(s, \tau, x(\tau)) d\tau) - f(s, 0, 0, 0)\| + \|f(s, 0, 0, 0)\|] ds \\
\leq\ & M_1(\|x_0\| + G_1) + M_1 L \int_0^t [r + \int_0^s \|k(s, \tau, x(\tau)) - k(s, \tau, 0) + k(s, \tau, 0)\| d\tau \\
& + \int_0^b \|h(s, \tau, x(\tau)) - h(s, \tau, 0) + h(s, \tau, 0)\| d\tau] ds + L_1 M_1 b \\
\leq\ & M_1(\|x_0\| + G_1) + M_1 \int_0^t [Lr + Lb(Kr + K_1) + Lb(Hr + H_1)] ds + L_1 M_1 b
\end{aligned}$$

$$\begin{aligned}&\leq M_1[\|x_0\| + G_1 + Lrb + LKrb^2 + LK_1b^2 + LHrb^2 + LH_1b^2 + L_1b]\\&= r. \end{aligned} \tag{5}$$

Thus, F maps E into itself and consequently $F \in C(E; E)$.

Now, we prove that F maps E into a precompact subset $F(E)$ of E. For this purpose, we first show that the set $E(t) = \{(Fx)(t) : x \in E\}$, $t \in J$ is precompact in X. Observe that

$$E(0) = \{(Fx)(0) : x \in E\} = \{x_0 - g(x) : x \in E_r, \quad x(0) + g(x) = x_0\}.$$

Therefore, according to hypothesis (H_4), $E(0)$ is precompact in X.

Let $t > 0$ be fixed. For an arbitrary $0 < \epsilon < t$, we define a mapping

$$\begin{aligned}(F_\epsilon x)(t) = R(t, 0)[x_0 - g(x)] + \int_0^{t-\epsilon} R(t, s)f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau,\\ \int_0^b h(s, \tau, x(\tau))d\tau)ds.\end{aligned} \tag{6}$$

Since $R(t, s)$ is compact operator for every $t, s \geq 0$, then the set $E_\epsilon(t) = \{(F_\epsilon x)(t) : x \in E\}$ is precompact in X for every $\epsilon > 0$. By using the equations (4), (6) and the hypotheses (H_1)–(H_4), we obtain

$$\begin{aligned}&\|(Fx)(t) - (F_\epsilon x)(t)\|\\&\leq M_1 \int_{t-\epsilon}^t [\|f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau, \int_0^b h(s, \tau, x(\tau))d\tau) - f(s, 0, 0, 0)\| + L_1]ds\\&\leq M_1 \int_{t-\epsilon}^t [Lr + Lb(Kr + K_1) + Lb(Hr + H_1) + L_1]ds\\&\leq M_1[Lrb + LKrb^2 + LK_1b^2 + LHrb^2 + LH_1b^2 + L_1b]\epsilon.\end{aligned} \tag{7}$$

This implies that there exist precompact sets arbitrary close to the set $E(t) = \{(Fx)(t) : x \in E\}$. Hence, the set $\{(Fx)(t) : x \in E\}$ is precompact in X.

Next, we show that $F(E)$ is an uniformly equicontinuous family of functions. Let $0 < s < t$. By using hypotheses hypotheses $(H_2), (H_3)$ and $(ii), (iii)$, we have

$$\begin{aligned}&\|(Fx)(t) - (Fx)(s)\|\\&\leq \|R(t, 0) - R(s, 0)\|(\|x_0\| + \|g(x)\|) + \int_0^s \|R(t, \tau) - R(s, \tau)\|\\&\quad \times \|f(\tau, x(\tau), \int_0^\tau k(\tau, \sigma, x(\sigma))d\sigma, \int_0^b h(\tau, \sigma, x(\sigma))d\sigma)\|d\tau\\&\quad + \int_s^t \|R(t, \tau)\|\|f(\tau, x(\tau), \int_0^\tau k(\tau, \sigma, x(\sigma))d\sigma, \int_0^b h(\tau, \sigma, x(\sigma))d\sigma)\|d\tau\\&\leq \|R(t, 0) - R(s, 0)\|(\|x_0\| + G_1) + M_1[Lr + L(Kr + K_1 + Hr + H_1)b + L_1](t - s)\\&\quad + \int_0^s \|R(t, \tau) - R(s, \tau)\|[Lr + L(Kr + K_1 + Hr + H_1)b + L_1]d\tau.\end{aligned} \tag{8}$$

Here we have proceeded as in the result (7). The right hand side of (8) is independent of $x \in E$ and tends to zero as $s \to t$ as a consequence of the continuity of $R(t,s)$ in the uniform operator topology for $t > 0$, which follows from the compactness of $R(t,s)$, $t - s > 0$. Therefore, $F(E)$ is equicontinuous family of functions. Thus by Arzela-Ascoli's theorem, $F(E)$ is precompact. Hence by the Schauder fixed point theorem, F has a fixed point in E and any fixed point of F is a mild solution of (1)–(2) on J.

4 Controllability Result

Controllability is one of the fundamental concepts in mathematical control theory and plays an important role in both deterministic and stochastic control systems. It is well known that the controllability of deterministic systems are widely used in many fields of science and technology. The controllability of nonlinear deterministic systems represented by equations in abstract spaces, whereas the stochastic control theory is a stochastic generalization of classical control theory. Such problems have been studied by several authors, see [1, 2, 4, 7, 8, 9] and the references cited therein. Now we will establish a set of sufficient conditions for the controllability of nonlinear mixed integrodifferential equation with control parameter of the form:

$$
\begin{aligned}
x^{'}(t) = A(t)\Big(x(t) + \int_0^t Q(t,s)x(s)ds\Big) + f(t, x(t), \int_0^t k(t,s,x(s))ds,& \\
\int_0^b h(t,s,x(s))ds) + (Bu)(t),\ t \in J& \qquad (9)\\
x(0) + g(x) = x_0,& \qquad (10)
\end{aligned}
$$

where the state $x(\cdot)$ takes values in the Banach space X and the control function $u(\cdot)$ is given in $L^2(J,U)$, a Banach space of admissible control functions with U as a Banach space. Here B is is a bounded linear operator from U into X. Then, for equations (9)–(10), there exists a mild solution of the following form

$$
\begin{aligned}
x(t) = R(t,0)[x_0 - g(x)] + \int_0^t R(t,s)\Big[f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau,& \\
\int_0^b h(s,\tau,x(\tau))d\tau) + (Bu)(s)\Big]ds,& \qquad (11)
\end{aligned}
$$

where the resolvent operator $R(t,s) \in B(X)$ for $t - s > 0$ and the functions f, g, k and h satisfy the conditions stated in Section 3.

DEFINITION. The system (9)–(10) is said to be nonlocally controllable on the interval J if, for every $x_1 \in X$, there exists a control $u \in L^2(J,U)$ such that the mild solution $x(\cdot)$ of the problem (9)–(10) satisfies $x(b) + g(x) = x_1$.

To establish the result, we need the following additional hypothesis.

(H_5) The operator W from $L^2(J,U)$ into X, defined by

$$Wu = \int_0^b R(b,s)(Bu)(s)ds,$$

has an induced inverse operator W^{-1} which takes values in $L^2(J,U)/kerW$, and there exist positive constants M_2, M_3 such that

$$\|B\| \leq M_2, \quad \|W^{-1}\| \leq M_3.$$

THEOREM 3. If the hypotheses (H_1)–(H_5) and conditions (ii), (iii) of Theorem 2 are satisfied, then the (9)–(10) is nonlocally controllable on J.

PROOF. Using hypothesis (H_5), for an arbitrary $x(\cdot)$, define the control

$$\begin{aligned} u(t) = W^{-1}\Big\{&x_1 - g(x) - R(b,0)(x_0 - g(x)) \\ &- \int_0^b R(b,s)\Big[f(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau, \int_0^b h(s,\tau,x(\tau))d\tau)\Big]ds\Big\}(t). \end{aligned} \tag{12}$$

Let

$$Z_0 = \{x \in Z : x(0) + g(x), \quad \|x\| \leq r_1, \quad \text{for} \quad t \in J\},$$

where the positive constant r_1 is given by

$$\begin{aligned} r_1 \geq M_1\Big[&\|x_0\| + G_1 + Lr_1b + LKr_1b^2 + LK_1b^2 + LHr_1b^2 \\ &+ LH_1b^2 + L_1b\Big](1 + M_1M_2M_3b) + M_1M_2M_3(\|x_1\| + G_1)b, \end{aligned}$$

with $(1 + M_1M_2M_3b)[M_1Lb + M_1LKb^2 + M_1LHb^2] < 1$. Then Z_0 is clearly a bounded, closed and convex subset of Z. Define a mapping $\Phi : Z_0 \to Z_0$ by

$$\begin{aligned} (\Phi x)(t) = R(t,0)(x_0 - g(x)) + \int_0^t R(t,s)\Big[&f(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau, \\ &\int_0^b h(s,\tau,x(\tau))d\tau) + (Bu)(s)\Big]ds. \end{aligned} \tag{13}$$

Now, we shall show that, when using control u, the operator Φ has a fixed point. This fixed point is then a mild solution of the system (9)–(10). Clearly, $x_1 - g(x) = (\Phi x)(b)$, which means that the control u steers the mixed integrodifferential system from the initial state x_0 to x_1 in time b provided we can obtain a fixed point of the nonlinear operator Φ. Using the definition of the control u in the equation (13), we get

$$\begin{aligned} (\Phi x)(t) = R(t,0)(x_0 - g(x)) &+ \int_0^t R(t,s)\Big[f(s,x(s),\int_0^s k(s,\tau,x(\tau))d\tau, \\ &\int_0^b h(s,\tau,x(\tau))d\tau)\Big]ds + \int_0^t R(t,s)BW^{-1}\Big[x_1 - g(x) - R(b,0)(x_0 - g(x)) \\ &- \int_0^b R(b,\theta)f(\theta,x(\theta),\int_0^\theta k(\theta,\tau,x(\tau))d\tau, \int_0^b h(\theta,\tau,x(\tau))d\tau)d\theta\Big](s)ds. \end{aligned} \tag{14}$$

Since all the functions involved in the definition of the operator are continuous, the operator Φ is continuous. For $x \in Z_0, t \in J$ and following steps as in the proof of

Theorem 2 in equation (5), from hypotheses $(H_1) - (H_5)$ and assumptions $(ii), (iii)$, we have

$$\|(\Phi x)(t)\| \leq M_1\Big[\|x_0\| + G_1 + Lr_1b + LKr_1b^2 + LK_1b^2 + LHr_1b^2 + LH_1b^2 + L_1b\Big](1 + M_1M_2M_3b) + M_1M_2M_3(\|x_1\| + G_1)b = r_1. \quad (15)$$

Thus, Φ maps Z_0 into itself and consequently $\Phi \in C(Z_0; Z_0)$.

Now, we prove that Φ maps Z_0 into a precompact subset $\Phi(Z_0)$ of Z_0. For this purpose, we first show that for every fixed $t \in J$, the set $Z_0(t) = \{(\Phi x)(t) : x \in Z_0\}$, is precompact in X. This is clear for $t = 0$ since $Z_0(0)$ is precompact by hypothesis (H_4).

Let $t > 0$ be fixed. For an arbitrary $0 < \epsilon < t$, we define a mapping

$$\begin{aligned}(\Phi_\epsilon x)(t) = R(t,0)(x_0 - g(x)) &+ \int_0^{t-\epsilon} R(t,s)\Big[f(s, x(s), \int_0^s k(s,\tau,x(\tau))d\tau, \\ &\int_0^b h(s,\tau,x(\tau))d\tau)\Big]ds \\ &+ \int_0^{t-\epsilon} R(t,s)BW^{-1}\Big[x_1 - g(x) - R(b,0)(x_0 - g(x)) - \int_0^b R(b,\theta)f(\theta, x(\theta), \\ &\int_0^\theta k(\theta,\tau,x(\tau))d\tau, \int_0^b h(\theta,\tau,x(\tau))d\tau)d\theta\Big](s)ds. \quad (16)\end{aligned}$$

Since $R(t,s)$ is compact operator for every $t, s \geq 0$, then the set $Z_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in Z_0\}$ is precompact in X for every $\epsilon > 0$. By using the equations (14), (16) and the hypotheses (H_1)–(H_4), and $(ii), (iii)$, we obtain

$$\begin{aligned}&\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| \\ &\leq M_1\Big[Lr_1 + LKr_1b + LK_1b + LHr_1b + LH_1b + L_1\Big]\epsilon + M_1M_2M_3\Big[\|x_1\| + G_1 \\ &\quad + M_1\big(\|x_0\| + G_1 + Lr_1b + LKr_1b^2 + LK_1b^2 + LHr_1b^2 + LH_1b^2 + L_1b\big)\Big]\epsilon, \quad (17)\end{aligned}$$

which implies that $Z_0(t)$ is totally bounded, that is, precompact in X.

Next, we show that $\Phi(Z_0)$ is an uniformly equicontinuous family of functions. Let $0 < s < t$. Following steps as in the proof of Theorem 2 in equation (8), from hypotheses hypotheses $(H_2), (H_3)$ and $(ii), (iii)$, we have

$$\begin{aligned}&\|(\Phi x)(t) - (\Phi x)(s)\| \\ &\leq \|R(t,0) - R(s,0)\|(\|x_0\| + G_1) + M_1[Lr + L(Kr + K_1 + Hr + H_1)b + L_1](t-s) \\ &\quad + \int_0^s \|R(t,\tau) - R(s,\tau)\|[Lr + L(Kr + K_1 + Hr + H_1)b + L_1]d\tau \\ &\quad + \int_0^s \|R(t,\tau) - R(s,\tau)\|M_2M_3\Big[\|x_1\| + G_1 + M_1\big(\|x_0\| + G_1 + (Lr_1 + LKr_1b \\ &\quad + LK_1b + LHr_1b + LH_1b + L_1)b\big)\Big]d\tau + M_1M_2M_3\Big[\|x_1\| + G_1 + M_1\big(\|x_0\| + G_1 \\ &\quad + (Lr_1 + LKr_1b + LK_1b + LHr_1b + LH_1b + L_1)b\big)\Big](t-s). \quad (18)\end{aligned}$$

Here we have proceeded as in the result (17). The right hand side of (18) is independent of $x \in Z_0$ and tends to zero as $s \to t$ as a consequence of the continuity of $R(t,s)$ in the uniform operator topology for $t > 0$, which follows from the compactness of $R(t,s)$, $t - s > 0$. Therefore, $\Phi(Z_0)$ is equicontinuous family of functions. Thus by Arzela-Ascoli's theorem, $\Phi(Z_0)$ is precompact. Hence by the Schauder fixed point theorem, Φ has a fixed point in Z_0 and any fixed point of Φ is a mild solution of (9)–(10) on J. Therefore, the system (9)–(10) is nonlocally controllable on J.

5 Example

In this section, we give an example to illustrate the usefulness of our main result. Let us consider the following partial integrodifferential equation of the form:

$$\begin{aligned} \frac{\partial}{\partial t}w(t,x) = a_0(t)\frac{\partial^2}{\partial x^2}[w(t,x) + \int_0^t \frac{1}{(1+t^2)(1+s^2)}w(s,x)ds] + \mu(t,x) \\ + w(t,x) + \int_0^t \frac{1}{(1+t^2)(1+s)}w(s,x)ds \\ + \int_0^1 \frac{1}{(1+t^2)(1+s)}[w^2(s,x) + sin(w^2(s,x))]ds \end{aligned} \tag{19}$$

$$w(t,0) = w(t,\pi) = 0 \tag{20}$$

$$w(0,x) + \int_0^1 \frac{1}{2}w(s,x)ds = w_0(x), \quad 0 \le t \le 1, \quad 0 \le x \le \pi, \tag{21}$$

where $w_0(x) \in X = L^2([0,\pi])$, $w_0(0) = w_0(\pi) = 0$ and the functions a_0 and $\mu : [0,1] \times (0,\pi) \to (0,\pi)$ are continuous on $0 \le t \le 1$. Let $X = L^2([0,\pi])$ and the operators $A(t)$ be defined by

$$A(t)z = a_0(t)z^{''}$$

with the domain $D(A) = \{z \in X : \quad z, \quad z^{''}$ are absolutely contunuous, $z^{''} \in X, \quad z(0) = z(1) = 0\}$, then $A(t)$ generates an evolution system and $R(t,s)$ can be deduced from the evolution systems [11, 12, 15] such that $R(t,s)$ is compact and $\|R(t,s)\| \le M_1 e^{\beta(t-s)}$ for some constants M_1 and β. On comparison of functions f and g with the problem (19)–(21), then by assumptions, we have

$$\|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)\| \le \{1 + \frac{\log 2}{(1+t^2)}[1 + 2\|x_1 + x_2\|]\}\|x_1 - x_2\|.$$

Also for $x_1, \quad x_2 \in C([0,1]$, the function

$$\begin{aligned} g(\lambda x_1 + (1-\lambda)x_2) &= \int_0^1 \frac{1}{2}(\lambda x_1 + (1-\lambda)x_2)(s)ds \\ &= \int_0^1 \frac{1}{2}\lambda x_1(s)ds + \int_0^1 \frac{1}{2}(1-\lambda)x_2(s)ds = \lambda g(x_1) + (1-\lambda)g(x_2) \end{aligned}$$

is convex.

Let $Bu : [0,1] \times X$ be defined by

$$(Bu)(t)x = \mu(t,x), \quad x \in (0,\pi).$$

With the choice of $A(t), \quad B$ and f, the equations (19)-(21) take the abstract form as (1)-(2).

Now, the linear operator W is given by

$$(Wu)x = \int_0^1 R(1,s)\mu(s,x)ds, \quad x \in (0,\pi).$$

Assume that this operator has a bounded invertible operator W^{-1} in $L^2([0,1],U)/kerW$. Further, all the other conditions stated in Theorem 2 and Theorem 3 are satisfied. Hence, the problem (19)-(21) has a mild solution on $[0,1]$ and the system (19)-(21) is controllable on $[0,1]$.

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References

[1] K. Balachandran and J. P. Dauer, Controllability of nonlinear systems in Banach spaces: a survey, J. Optim. Theory Appl., 115(2002), 7–28.

[2] K. Balachandran and J. Y. Park, Existence of solutions and controllability of nonlinear integrodifferential systems in Banach spaces, Mathematical Problems in Engineering 2003, 2(2003), 65–79.

[3] K. Balachandran, R. R. Kumar, Existence of solutions of integrodifferential evolution with time varying delays, AMEN, 7(2007), 1–8.

[4] K. Balachandran and R. Sakthivel, Controllability of integrodifferential systems in Banach spaces, Appl. Math. Comput., 118(2001), 63–71.

[5] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl., 162(1991), 494-505.

[6] L. Byszewski and H. Akca, Existence of solutions of a semilinear functional-differential evolution nonlocal problem, Nonlinear Anal., 34(1998), 65-72.

[7] D. N. Chalishajar, Controllability of mixed Volterra-Fredholm type integrodifferential systems in Banach spaces, Journal of the Franklin Institute, 344(2007), 12–21.

[8] Y. K. Chang and L. M. Qi, Controllability of nonautonomous semilinear integrodifferential inclusions in Banach spaces, Mathematical Inequalities and Applications, 10(2)(2007), 343–350.

[9] Y. K. Chang, J. J. Nieto and W. S. Li, Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces, J. Optim. Theory Appl., 142(2009), 267–273.

[10] K. Deng, Exponential delay of solutions of semilinear parabolic equations with nonlocal initial conditions, J. Math. Anal. Appl., 179(1993), 630–637.

[11] R. Grimmer, Resolvent operators for integral equations in Banach space, Transactions of the AMS, 273(1982), 333–349.

[12] R. Grimmer and J. H. Liu, Integrated semigroups and integrodifferential equations, Semigroup Forum, 48(1994), 79–95.

[13] J. H. Liu and K. Ezzinbi, Non-autonomous integrodifferential equations with nonlocal conditions, J. Integral Eq. Appl., 15(1)(2003), 79–93.

[14] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, New York, Springer-Verlag, 1983.

[15] J. Pruss, On resolvent operators for linear integrodifferential equations of Volterra type, J. Integral Equations, 5(1983), 211–236.

[16] H. L. Tidke and M. B. Dhakne, On global existence of solutions of abstract nonlinear mixed integrodifferential equation with nonlocal condition, Communications on Applied Nonlinear Analysis, 16(1)(2009), 49–60.

[17] H. L. Tidke, Existence of global solutions to nonlinear mixed Volterra-Fredholm integrodifferential equation with nonlocal condition, Electronic Journal of Differential Equations, 2009(2009), No. 55, 1–7.

[18] G. Teschl, Nonlinear Functional Analysis, Vienna, Austria, 2001.

[19] Z. M. Yan, On solutions of semilinear evolution integrodifferential equations with nonlocal conditions, Tamkang J. Math., 40(3)(2009), 257–269.

[20] Z. M. Yan, Nonlocal problems for delay integrodifferential equations in Banach spaces, Differential Equations and Applications, in Press.