Applied Mathematics E-Notes, 11(2011), 73-81 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Generators For Nonlinear Three Term Recurrences Involving The Greatest Integer Function*

Yen Chih Chang†, Gen-qiang Wang‡

Received 1 February 2011

Abstract

We show that all real solutions of the nonlinear recurrence relation $\varphi_{i+1} + \varphi_{i-1} = [\varphi_i]$ can be expressed explicitly in terms of solutions u and ε that satisfy $(u_0, u_1) = (1, 0)$ and $(\varepsilon_0, \varepsilon_1) = (s, 0)$ where $s \in [0, 1/2]$. As applications, we are able to show the translation symmetry between solutions and that all solutions have prime periods $1, 4$, 6 or 12.

1 Introduction

For linear homogeneous recurrence relations such as

$$\varphi_{i+1} + \varphi_{i-1} = \varphi_i,\ i \in \mathbf{Z}. \tag{1}$$

where Z is the set of integers, it is well known that all its real (or complex) solutions can be explicitly given. Indeed, we may find the solution $u = \{u_i\}_{i=\mathbf{Z}}$ that satisfies $(u_0, u_1) = (1, 0)$ and the solution $v = \{v_i\}_{i=\mathbf{Z}}$ that satisfies $(v_0, v_1) = (0, 1)$ and then any other solution φ of (1) is of the form

$$\varphi = \varphi_0 u + \varphi_1 v. \tag{2}$$

The solutions u and v can further be determined. Indeed, the characteristic equation of (1) is

$$x^2 - x + 1 = 0,$$

which is a cyclotomic polynomial of order 6 with roots $\lambda_\pm = \cos(\frac{\pi}{3}) \pm i \sin(\frac{\pi}{3})$. Then the sequences u and v can be determined to be

$$u = \left\{ \cos\left(\frac{k\pi}{3}\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{k\pi}{3}\right) \right\}_{k\in\mathbf{Z}} \tag{3}$$

and

$$v = \left\{ \frac{2}{\sqrt{3}} \sin\left(\frac{k\pi}{3}\right) \right\}_{k\in\mathbf{Z}} \tag{4}$$

*Mathematics Subject Classifications: 11B37, 11B50.

†Library, Tsing Hua University, Hsinchu, Taiwan 30043, R. O. China

‡Department of Computer Science, Guangdong Polytechnic Normal University, Guangzhou, Guangdong 510663, P. R. China

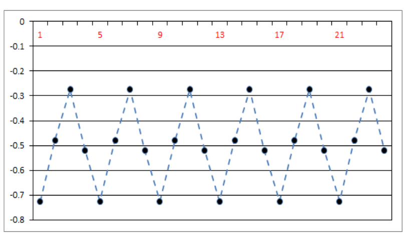
respectively (see e.g. [1]).

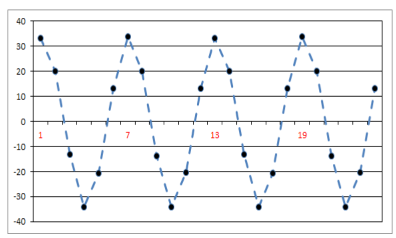
For nonlinear recurrence relations, however, there are no unified theories for finding the explicit forms of their solutions and each relation has to be handled in a unique manner. Therefore it is of great interest to show that some nonlinear recurrences may admit general solutions and from the corresponding explicit expressions interesting consequences can be derived.

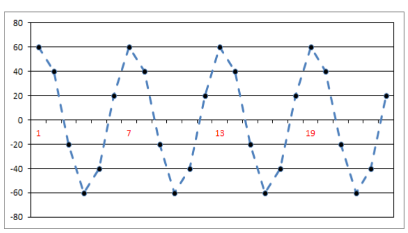
In this note, we show that for a related nonlinear three term recurrence relation[1]

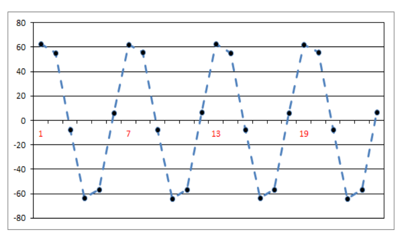
$$\varphi_{i+1} + \varphi_{i-1} = [\varphi_i]\,,\ i \in \mathbf{Z}, \tag{5}$$

where $[x]$ is the integral part of the real number x, all its real solutions can be given in explicit forms. Indeed, we may roughly see this fact as follows. First, adding an integer solution to the recurrence (1) to any solution to the nonlinear recurrence (5) gives a new solution to the nonlinear recurrence. This immediately lets one reduce to the case in which φ_0 and φ_1 are 0. Moreover, replacing φ_1 by 0 does not change the subsequent terms of the sequence. Thus one need only see what happens to the recurrence when it is seeded with $\varphi_0 = s$ and $\varphi_1 = 0$ for some s in $[0, 1)$, and this can be done by carefully go through the solutions sequences one by one. Although the idea is quite simple, there are still many details and fine adjustments. Indeed, we show that it is better to use $s \in [0, 1/2]$, and that as applications, we may, in a relatively easy manner, show some of the symmetry (under translation) and periodic properties of the real solutions of (5) (see the following Figures in which numerical simulations of solutions of (5) are shown).









Before deriving our results, let us first go through some elementary concepts and properties of our nonlinear equation (5).

[1] This nonlinear recurrence may be regarded as the steady state equation for an infinite linear array of artificial neuron pools obeying the update relation $\varphi_i^{(t+1)} - \varphi_i^{(t)} = \varphi_{i-1}^{(t)} + \varphi_{i+1}^{(t)} - \left[\varphi_i^{(t)}\right]$, $t = 0, 1, ...; i \in \mathbf{Z}$.

First, for any $x \in \mathbf{R}$, we denote the fractional part of x as

$$\langle x \rangle = x - [x]\,.$$

Hence, $\langle x \rangle \in [0,1)$, and if α is an integer and $\varepsilon \in [0,1)$, then $[\alpha+\varepsilon] = [\alpha] = \alpha$ and $[\alpha-\varepsilon] = [\alpha-1] = \alpha-1$, and if β, γ are two integers, then $[\beta]+[\gamma] = [\beta+\gamma] = \beta+\gamma$.

A real sequence $\psi = \{\psi_i\}_{i\in\mathbf{Z}}$ is said to be **integral** if each ψ_i is an integer. A **translation** of ψ is a sequence $E^j\psi$, where $j \in \mathbf{Z}$, defined by

$$(E^j\psi)_m = \psi_{m-j},\ m \in \mathbf{Z},$$

(in particular, $E^0\psi = \psi$).

A (scalar or vector) sequence $\psi = \{\psi_m\}_{m\in\mathbf{Z}}$ is said to be **periodic** if there is a positive integer τ such that $\psi_{i+\tau} = \psi_i$ for all $i \in \mathbf{Z}$. The positive integer τ is called a period of ψ. If ψ is periodic, then among all periods of ψ, there is the least one, which we will denote by $\Omega\psi$. If $\Omega\psi = \omega$, then ψ is said to be ω-periodic (or said to have **prime period** ω). For an ω-periodic sequence ψ, it is clearly determined completely by one of its cycles of the form

$$\psi^{[\alpha]} = (\psi_\alpha, \psi_{\alpha+1}, ..., \psi_{\alpha+\omega-1}).$$

Three elementary facts about periodic sequences are: (i) the prime period of a real sequence ψ is a factor of its periods, (ii) a sequence and its translation have the same prime period, and (iii) if ζ and ξ are two real sequences with prime periods m and n respectively, then the least common multiple of m and n is a period of $\zeta+\xi$.

2 General Solutions

A solution of (5) is a real sequence $\varphi = \{\varphi_i\}_{i\in\mathbf{Z}}$ which renders (5) into an identity after substitution. Since (5) can be rewritten as

$$\varphi_{i+1} = [\varphi_i] - \varphi_{i-1}, \tag{6}$$

or

$$\varphi_{i-1} = [\varphi_i] - \varphi_{i+1}, \tag{7}$$

therefore, we may see that a solution φ of (5) is uniquely determined by any two of its consecutive terms $(\varphi_k, \varphi_{k+1})$. In particular, we may easily see that φ is an **integral solution** of (5) if, and only if, for any $k \in \mathbf{Z}$, φ_k and φ_{k+1} are integers. By means of the properties of the Gauss function, another elementary fact is also easy to see.

Lemma 1. φ is an integral solution of (5) if, and only if, it is an integral solution of (1).

In view of Lemma 1 and our previous discussions about (1), we see that every integral solution φ of (5) is of the form

$$\varphi = \varphi_0 u + \varphi_1 v. \tag{8}$$

Although u and v were already been given by (3) and (4), we may also utilize the facts that u and v are 6-periodic and derive them directly from (5). Indeed, by direct computation, we may easily see that

$$u^{[0]} = (1, 0, -1, -1, 0, 1) \tag{9}$$

and

$$v^{[0]} = (0, 1, 1, 0, -1, -1).$$

By means of these cycles, we may observe the interesting fact that

$$v = E^2 u.$$

Furthermore, by (5), we see that $\varphi = \{0\}$ if, and only if, $(\varphi_0, \varphi_1) = (0, 0)$.

Next, let us call a solution $\varphi = \{\varphi_i\}_{i\in\mathbf{Z}}$ of (5) **fractional** if $\varphi_0, \varphi_1 \in [0, 1)$.

We assert that all fractional solutions of (5) can be 'generated' by an 1-parameter family of solutions of (5). To this end, let $s \in [0, 1/2]$ and let $\varepsilon(s)$ be the solution of (5) determined by the initial condition

$$(\varepsilon_0(s), \varepsilon_1(s)) = (s, 0),\ \ s \in [0, 1/2].$$

Lemma 2. For $s \in [0, 1/2]$, let $\varepsilon(s)$ be the fractional solution of (5) with $(\varepsilon_0(s), \varepsilon_1(s)) = (s, 0)$. Then,

(i) $\varepsilon(0) = \{0\}_{i\in\mathbf{Z}}$,

(ii) $\varepsilon(1/2)$ is 6-periodic and $\varepsilon^{[0]} = (1/2, 0, -1/2, -1, -1/2, 0)$, and

(iii) if $s \in (0, 1/2)$, then $\varepsilon(s)$ is 12-periodic and

$$\varepsilon^{[12]} = (s, 0, -s, -1, -1 + s, 0, s', 0, -s', -1, -1 + s', 0)$$

where $s' = 1 - s$.

Proof: Indeed, by (5), we may calculate $\varepsilon(0)$ or $\varepsilon(1/2)$ directly and check (i) as well as (ii).

Now, we consider case (iii) where $s \in (0, 1/2)$. By (5), $\varepsilon(s)$ is periodic with period 12. Next, we may check that $1, 2, 3, 4, 6$ are not periods of $\varepsilon(s)$. Indeed, if 6 is a period of $\varepsilon(s)$, then $s = \varepsilon_0(s) = \varepsilon_6(s) = 1 - s$ so that we obtain the contradiction $s = 1/2$. The other three cases can be handled in similar manners. The proof is complete.

For $i \in \{0, 6\}$ and $j \in \{1, 7\}$, by direct verification, it is easily shown that $E^i\epsilon\,(s) + E^j\epsilon\,(t)$ is a fractional solution of (5). By Lemma 2, 12 is a period of $\varepsilon(s)$ and $\varepsilon(t)$ and hence is also a period of $E^i\varepsilon\,(s) + E^j\varepsilon\,(t)$. Conversely, any fractional solution of (5) is of the form $E^i\epsilon\,(s) + E^j\epsilon\,(t)$ for some $i \in \{0, 6\}, j \in \{1, 7\}$ and $s, t \in [0, 1/2]$. Indeed, let φ be a fractional solution of (5). Then $\varphi_0, \varphi_1 \in [0, 1)$. We may check that:

(i) if $(\varphi_0, \varphi_1) \in [0, 1/2] \times [0, 1/2]$, then

$$\varphi = E^0\varepsilon(\varphi_0) + E^1\varepsilon(\varphi_1), \tag{10}$$

(ii) if $(\varphi_0, \varphi_1) \in [0, 1/2) \times (1/2, 1)$, then

$$\varphi = E^0\varepsilon(\varphi_0) + E^7\varepsilon(1 - \varphi_1), \tag{11}$$

(iii) if $(\varphi_0, \varphi_1) \in (1/2, 1) \times [0, 1/2)$, then

$$\varphi = E^6 \varepsilon(1 - \varphi_0) + E^1 \varepsilon(\varphi_1), \tag{12}$$

(iii) if $(\varphi_0, \varphi_1) \in [1/2, 1) \times [1/2, 1) \setminus \{(1/2, 1/2)\}$, then

$$\varphi = E^6 \varepsilon(1 - \varphi_0) + E^7 \varepsilon(1 - \varphi_1). \tag{13}$$

For instance, to show (10), we only need to see from Lemma 2 that

$$\left(E^0 \varepsilon(\varphi_0) + E^1 \varepsilon(\varphi_1)\right)_0 = ((\varphi_0, 0, ..., 0) + (0, \varphi_1, ..., -\varphi_1))_0 = \varphi_0,$$

and

$$\left(E^0 \varepsilon(\varphi_0) + E^1 \varepsilon(\varphi_1)\right)_1 = ((\varphi_0, 0, ..., 0) + (0, \varphi_1, ..., -\varphi_1))_1 = \varphi_1.$$

To simplify the above expressions, let us define a mapping

$$\Gamma(s, t) = (\Gamma_1(s, t), \Gamma_2(s, t)) = \begin{cases} (0, 1) & \text{if } (s, t) \in [0, 1/2] \times [0, 1/2], \\ (0, 7) & \text{if } (s, t) \in [0, 1/2) \times (1/2, 1), \\ (6, 1) & \text{if } (s, t) \in (1/2, 1) \times [0, 1/2), \\ (6, 7) & \text{if } (s, t) \in [1/2, 1) \times [1/2, 1) \setminus \{(1/2, 1/2)\}, \end{cases} \tag{14}$$

and define

$$x^* = \begin{cases} x & \text{if } x \in [0, 1/2], \\ 1 - x & \text{if } x \in (1/2, 1). \end{cases} \tag{15}$$

Then we have the following conclusion.

Lemma 3. For $i \in \{0, 6\}$ and $j \in \{1, 7\}$, $E^i \varepsilon(s) + E^j \varepsilon(t)$ is a fractional solution of (5) (with period 12). Conversely, any fractional solution φ of (5) is equal to

$$E^{\Gamma_1(\varphi_0, \varphi_1)} \varepsilon(\varphi_0^*) + E^{\Gamma_2(\varphi_0, \varphi_1)} \varepsilon(\varphi_1^*).$$

Lemma 4. Any solution ζ of (5) is of the form $\psi + \epsilon$ where ψ is the integral solution of (5) determined by $(\psi_0, \psi_1) = ([\zeta_0], [\zeta_1])$ and ϵ is the fractional solution of (5) determined by $(\epsilon_0, \epsilon_1) = (\langle \zeta_0 \rangle, \langle \zeta_1 \rangle)$.

The proof is rather easy. Indeed, it may be verified directly that $\psi + \epsilon$ is a solution of (5). Since $(\psi + \epsilon)_0 = \zeta_0$ and $(\psi + \epsilon)_1 = \zeta_1$, by the uniqueness property of solutions of (5), we see that $\zeta = \psi + \epsilon$.

From the previous Lemmas, we have the following main result.

Theorem 1. For any $\alpha, \beta \in \mathbf{Z}$ and $i \in \{0, 6\}, j \in \{1, 7\}$, $\alpha u + \beta \left(E^2 u\right) + E^i \varepsilon(s) + E^j \varepsilon(t)$ is a solution of (5). Conversely, any real solution φ of (5) is of the form

$$\varphi = \alpha u + \beta E^2 u + E^i \varepsilon(s) + E^j \varepsilon(t) \tag{16}$$

where

$$\alpha = [\varphi_0], \beta = [\varphi_1], s = \langle \varphi_0 \rangle^*, t = \langle \varphi_1 \rangle^*, (i, j) = \Gamma(\langle \varphi_0 \rangle, \langle \varphi_1 \rangle) \tag{17}$$

(where $u^{[0]}$ is given by (9), $\varepsilon(s)$ by Lemma 2, Γ by (14) and x^* by (15)).

Indeed, for any $\alpha, \beta \in \mathbf{Z}$ and $i \in \{0, 6\}, j \in \{1, 7\}$, $\alpha u + \beta \left(E^2 u\right)$ is an integral solution and $E^i \varepsilon(s) + E^j \varepsilon(t)$ is a fractional solution of (5). Hence their sum by Lemma 4 is a solution of (5). Conversely, if φ is a solution of (5), then by Lemma 4, $\varphi = \psi + \epsilon$ where ψ is the integral solution of (5) determined by $(\psi_0, \psi_1) = ([\varphi_0], [\varphi_1])$ and hence by Lemma 1 and (8), $\psi = [\varphi_0] u + [\varphi_1] E^2 u$; while ϵ is the fractional solution

$$\varphi - [\varphi_0] u + [\varphi_1] E^2 u = E^i \varepsilon(s) + E^j \varepsilon(t)$$

of (5) determined by

$$(\epsilon_0, \epsilon_1) = \left(\left\langle \left(\varphi - [\varphi_0] u + [\varphi_1] E^2 u\right)_0 \right\rangle, \left\langle \left(\varphi - [\varphi_0] u + [\varphi_1] E^2 u\right)_1 \right\rangle\right) = (\langle\varphi_0\rangle, \langle\varphi_1\rangle)$$

and hence by Lemma 3, $s = \langle\varphi_0\rangle^*, t = \langle\varphi_1\rangle^*, (i, j) = \Gamma(\langle\varphi_0\rangle, \langle\varphi_1\rangle)$.

As direct consequences, we may calculate several specific solutions: Let φ be the solution of (5) with

$$([\varphi_0], [\varphi_1], \langle\varphi_0\rangle, \langle\varphi_1\rangle) = (\alpha, \beta, s, t).$$

(i) If $(\alpha, \beta, s, t) = (0, 0, 0, 0)$, then

$$(\varphi_0, \varphi_1, ..., \varphi_{12}) = (0, 0, ..., 0).$$

(ii) If $(\alpha, \beta, s, t) = (-1, -1, 1/2, 1/2)$, then

$$(\varphi_0, \varphi_1, ..., \varphi_{12}) = (-1/2, -1/2, ..., -1/2). \tag{18}$$

(iii) If $(s, t) \in (0, 1/2) \times \{0\}$, then

$$\begin{aligned} & (\varphi_0, \varphi_1, ..., \varphi_{12}) \\ = \; & (\alpha, \beta, \beta - \alpha, -\alpha, -\beta, \alpha - \beta, \alpha, \beta, \beta - \alpha, -\alpha, -\beta, \alpha - \beta) \\ & + (s, 0, -s, -1, -1 + s, 0, 1 - s, 0, -1 + s, -1, -s, 0). \end{aligned} \tag{19}$$

(iv) If $(s, t) \in \{0\} \times (0, 1/2)$, then

$$\begin{aligned} & (\varphi_0, \varphi_1, ..., \varphi_{12}) \\ = \; & (\alpha, \beta, \beta - \alpha, -\alpha, -\beta, \alpha - \beta, \alpha, \beta, \beta - \alpha, -\alpha, -\beta, \alpha - \beta) \\ & + (0, t, 0, -t, -1, -1 + t, 0, 1 - t, 0, -1 + t, -1, -t). \end{aligned} \tag{20}$$

(v) If $(s, t) \in (0, 1/2] \times (0, 1/2]$ or $(1/2, 1) \times (0, 1/2)$, then

$$\begin{aligned} & (\varphi_0, \varphi_1, ..., \varphi_{12}) \\ = \; & (\alpha, \beta, \beta - \alpha, -\alpha, -\beta, \alpha - \beta, \alpha, \beta, \beta - \alpha, -\alpha, -\beta, \alpha - \beta) \\ & + (s, t, -s, -1 - t, -2 + s, -1 + t, 1 - s, 1 - t, -1 + s, -2 + t, -1 - s, -t) \end{aligned} \tag{21}$$

We may also obtain several symmetry properties (under translation) of (5) as follows.

Corollary 1. Let φ and ζ be solutions of (5). Then $\varphi = E^6 \zeta$ provided one of the following conditions hold:

(i) $[\varphi_0] = [\zeta_0]$, $[\varphi_1] = [\zeta_1]$, $\langle\varphi_0\rangle = 1 - \langle\zeta_0\rangle \in (1/2, 1)$ and $\langle\varphi_1\rangle = 0 = \langle\zeta_1\rangle$;

(ii) $[\varphi_0] = [\zeta_0]$, $[\varphi_1] = [\zeta_1]$, $\langle\varphi_0\rangle = 0 = \langle\zeta_0\rangle$ and $\langle\varphi_1\rangle = 1 - \langle\zeta_1\rangle \in (1/2, 1)$;

(iii) $[\varphi_0] = [\zeta_0]$, $[\varphi_1] = [\zeta_1]$, $\langle\varphi_0\rangle = 1 - \langle\zeta_0\rangle \in [1/2, 1)$ and $\langle\varphi_1\rangle = 1 - \langle\zeta_1\rangle \in [1/2, 1)$;

(iv) $[\varphi_0] = [\zeta_0]$, $[\varphi_1] = [\zeta_1]$, $\langle\varphi_0\rangle = \langle\zeta_0\rangle \in (0, 1/2)$ and $\langle\varphi_1\rangle = 1 - \langle\zeta_1\rangle \in (1/2, 1)$.

For instance, to show (i), we only need to see that

$$\begin{aligned} E^6\zeta &= E^6\left(\alpha u + \beta E^2 u + E^0\varepsilon\left(\langle\zeta_0\rangle\right) + E^1\varepsilon\left(0\right)\right) \\ &= \alpha u + \beta E^2 u + E^6\varepsilon\left(\langle\zeta_0\rangle\right) + E^7\varepsilon(0) \\ &= \varphi. \end{aligned}$$

The other cases can be checked in similar manners.

We remark that the sequences φ and ζ in the above result have the same prime period since φ is a translation of ζ. This fact will be used in the next section.

3 Periodicity

Let ψ be a solution of (5). If

$$(\langle\psi_0\rangle, \langle\psi_1\rangle) \in \{\{0\} \times (0,1)\} \cup \{(0,1) \times \{0\}\} \setminus \{(0, 1/2), (1/2, 0)\},$$

then ψ is 12-periodic. Indeed, by Theorem 1, ψ is periodic with period 12. For convenience, let

$$([\psi_0], [\psi_1], \langle\psi_0\rangle, \langle\psi_1\rangle) = (\alpha, \beta, s, t). \tag{22}$$

By the symmetry properties stated in Corollary 1, we only have to show the cases where $(s,t) \in (0,1/2) \times \{0\}$ and $(s,t) \in \{0\} \times (0,1/2)$. In the former case, by (19), if $\psi_6 = \alpha + 1 - s = \alpha + s = \psi_0$, then $s = 1/2$ that is a contradiction. Thus 6 is not a period of ψ. We may similarly show that $1, 2, 3$ are not periods of ψ, If $\psi_0 = \alpha + s = -\beta - 1 + s = \psi_4$ and $\psi_1 = \beta = \alpha - \beta = \psi_5$, then $(\alpha, \beta) = (-2/3, -1/3)$ that is also contradictory. Hence 4 is not a period of ψ. Thus ψ is 12-periodic. The later case can be handled in similar manners. The proof is complete.

Next, let ψ be a solution of (5) with (22). Suppose

$$(s,t) \in \{(1/2, 0), (0, 1/2), (1/2, 1/2)\}.$$

If $(\alpha, \beta, s, t) = (-1, -1, 1/2, 1/2)$, then ψ is 1-periodic (and $\psi = \{-1/2\}$); otherwise, ψ is 6-periodic. Indeed, the former statement follows from (18). As for the later statement, note that $\Gamma(0, 1/2) = \Gamma(1/2, 1/2) = \Gamma(1/2, 0) = (0, 1)$. Suppose $(\alpha, \beta) \neq (-1, -1)$. Then by (19) or (20), we see that 6 is a period of ψ since $(\langle\psi_6\rangle, \langle\psi_7\rangle) = \left(\langle\psi_0\rangle^*, \langle\psi_1\rangle^*\right) = (\langle\psi_0\rangle, \langle\psi_1\rangle)$. We first consider $(s,t) = (1/2, 1/2)$. Suppose $\psi_0 = \alpha + 1/2 = -\alpha - 3/2 = \psi_3$ and $\psi_1 = \beta + 1/2 = -\beta - 3/2$. Then $(\alpha, \beta) = (-1, -1)$ which is a contradiction and hence ψ cannot be 1- nor 3-periodic. If 2 is a period, then $\psi_0 = \alpha + 1/2 = -\beta - \alpha - 1/2 = \psi_2$ and $\psi_1 = \beta + 1/2 = -\alpha - 1 - 1/2 = \psi_3$ which lead to $(\alpha, \beta) = (-1, -1)$ which is contradictory to our assumption. Therefore,

ψ is 6-periodic. The other two cases can be proved in similar manners. The proof is complete.

Next, let ψ be a solution of (5) with (22). Suppose $(s,t) \in (0,1) \times (0,1) \setminus \{(1/2,1/2)\}$. If $(\alpha,\beta) = (-1,-1)$, then $\Omega\psi = 4$; otherwise, $\Omega\psi = 12$. Indeed, in view of Corollary 1, we only have to consider the case where $(s,t) \in \{(0,1/2] \times (0,1/2]\} \setminus \{(1/2,1/2)\}$ and $(1/2,1) \times (0,1/2)$. However, by (21), we may just show the case where $(s,t) \in \{(0,1/2] \times (0,1/2]\} \setminus \{(1/2,1/2)\}$. We first show that ψ is either 4- or 12-periodic. To this end, we first apply Theorem 1 to obtain

$$\psi = \alpha u + \beta \left(E^2 u\right) + E^0 \epsilon(s) + E^1 \epsilon(t), \tag{23}$$

which has the period 12. Thus, we have

$$(\psi_4, \psi_5, \psi_6, \psi_7) = (-\beta, \alpha - \beta, \alpha, \beta) + (-2 + s, -1 + t, 1 - s, 1 - t). \tag{24}$$

If

$$\begin{aligned} \psi_0 &= \alpha + s = \alpha + 1 - s = \psi_6, \\ \psi_1 &= \beta + t = \beta + 1 - t = \psi_7, \end{aligned}$$

then $(s,t) = (1/2,1/2)$ which is a contradiction. Hence ψ cannot be 1-, 2-, 3-, 6-periodic, that is ψ is either 4- or 12-periodic. Next, suppose $\Omega\zeta = 4$. Then by (24), we have $\psi_4 = -\beta - 2 + s = \alpha + s = \psi_0$ and $\psi_5 = \alpha - \beta - 1 + t = \beta + t = \psi_1$ which is equivalent to $(\alpha,\beta) = (-1,-1)$.

We summarize the above discussions in the following result which provide the prime periods of all the solutions of (5).

Theorem 2. For any solution ψ of (5) with $(\alpha,\beta,s,t) = ([\psi_0], [\psi_1], \langle\psi_0\rangle, \langle\psi_1\rangle)$.

(i) If $(\alpha,\beta,s,t) = (0,0,0,0)$ or $(-1,-1,1/2,1/2)$, then $\psi = \{0\}$ or $\{-1/2\}$ respectively.

(ii) If $(s,t) \in \{(0,1) \times (0,1)\} \setminus \{(1/2,1/2)\}$ and $(\alpha,\beta) = (-1,-1)$, then ψ is 4-periodic.

(iii) If $(s,t) \in \{(1/2,0), (0,1/2), (1/2,1/2)\}$ and $(\alpha,\beta,s,t) \neq (-1,-1,1/2,1/2)$, then ψ is 6-periodic.

(iv) If $(s,t) \in \{(0,1) \times \{0\}\} \cup \{\{0\} \times (0,1)\} \setminus \{(1/2,0), (0,1/2)\}$, then ψ is 12-periodic.

(v) If $(s,t) \in \{(0,1) \times (0,1)\} \setminus \{(1/2,1/2)\}$ and $(\alpha,\beta) \neq (-1,-1)$, then ψ is 12-periodic.

4 Conclusions

For the nonlinear recurrence relation

$$\varphi_{i+1} + \varphi_{i-1} = [\varphi_i], \; i \in \mathbf{Z}, \tag{25}$$

all its real solutions can be expressed in the form

$$\varphi = \alpha u + \beta E^2 u + E^i \varepsilon(s) + E^j \varepsilon(t),$$

where u and $\varepsilon(x)$ are solutions of (25) given by (9) and Lemma 2 respectively, and α, β, s, t, i and j are given by (17). By means of this general solution, we are able to identify a large number of solutions of (25) as translations of some other solutions as in Corollary 1, and then show that all solutions of (25) have prime periods $1, 4, 6$ and 12.

Acknowledgment. We are grateful for all the help from Professor Sui Sun Cheng during the preparation of this paper. The second author is supported by the Natural Science Foundation of Guangdong Province of P. R. China (under grant number 9151008002000012).

References

[1] F. S. Roberts, Applied Combinatorics, Prentice Hall Englewood Cliffs, New Jersey, 1984.