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Semi-Convergence Of The Local Hermitian And Skew-Hermitian Splitting Iteration Methods For Singular Generalized Saddle Point Problems*

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Abstract

In this paper, the local Hermitian and skew-Hermitian splitting (LHSS) iteration method and the modified LHSS (MLHSS) iteration method for solving singular generalized saddle point problems were investigated. When A is non-Hermitian positive definite and the Hermitian part of A is dominant, the semi-convergence conditions are given, which generalize some results of Jiang and Cao for the nonsingular generalized saddle point problems to the singular generalized saddle point problems.

1 Introduction

We consider the following 2×2 block linear systems of the form:

$$\mathcal{A}u = \begin{pmatrix} A & B \\ B^* & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} = b, \tag{1}$$

where $A \in \mathbb{C}^{m\times m}$ is a positive definite matrix, $B \in \mathbb{C}^{m\times n}$ with $\operatorname{rank} B = r$ and $m > n$, $f \in \mathbb{C}^m$ and $g \in \mathbb{C}^n$ are two given vectors, and B^* is the conjugate transpose of the matrix B. When $r = n$, the coefficient matrix $\mathcal{A}$ is nonsingular and the linear systems (1) have a unique solution. When $r < n$, the coefficient matrix $\mathcal{A}$ is singular, under this case, we assume that the linear systems (1) are consistent, i.e., $b \in \mathcal{R}(\mathcal{A})$, the range of $\mathcal{A}$. When A is non-Hermitian positive definite or Hermitian positive definite, respectively, the linear systems (1) are referred to as generalized saddle point problems or saddle point problems, which are important and arise in a large number of scientific and engineering applications, such as the field of computational fluid dynamics [2], constrained and weighted least squares [3], interior point methods in constrained

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optimization [4], mixed finite element approximations of elliptic partial differential equations [5]. See [1] for a comprehensive survey.

In order to solve the linear system (1) with iterative method, based on the matrix splitting, the coefficient matrix $\mathcal{A}$ can be written as

$$\mathcal{A} = M - N, \tag{2}$$

where $M \in \mathbb{C}^{(m+n)\times(m+n)}$ is a nonsingular matrix. Then the associated stationary iterative scheme for solving the systems (1) can be described as follows

$$u_{k+1} = Tu_k + c, \; k = 0, 1, 2, \cdots \tag{3}$$

where $u_0 \in \mathbb{C}^{m+n}$ is an initial guess, $c = M^{-1}b$, and $T = M^{-1}N$ is the iteration matrix. It is well known that for nonsingular (singular) systems the iterative method (3) is convergent (semi-convergent) if and only if T is a convergent (semi-convergent) matrix. On the semi-convergence of the iterative methods for general singular linear systems, one can see [12, 13, 29, 30, 32, 33, 34].

In recent years, when A is Hermitian positive definite and B is of full column rank, a large amount of work have been developed to solve the linear system (1), such as Uzawa type methods [6, 9, 14, 15, 16, 25, 27, 28], HSS iteration methods [17, 18, 19, 20, 21], preconditioned Krylov subspace iteration methods [7, 8]. When A is non-Hermitian positive definite, B is of full column rank, iterative methods have also been studied in [10, 11, 22, 23, 26, 31]. For a broad overview of the numerical solution of linear systems (1), one can see [1] for more details. In most cases, the matrix B is of full column rank in scientific computing and engineering applications, but not always. If $r < n$, the linear systems (1) become the singular saddle point problems. When the linear systems (1) are consistent, Zheng, Bai and Yang [24] show that the GSOR method is semi-convergent with A symmetric positive definite.

Recently, Jiang and Cao [31] presented a local Hermitian and skew-Hermitian splitting (LHSS) iteration method and a modified LHSS (MLHSS) iteration method for solving nonsingular systems (1). When A is non-Hermitian positive definite and the Hermitian part of A is dominant, some convergence conditions for these methods are given under suitable preconditioners.

In this paper, we further investigate the LHSS and MLHSS iteration methods for solving singular linear systems (1). When A is non-Hermitian positive definite and the Hermitian part of A is dominant, the semi-convergence conditions are proposed, which generalize some results of Jiang and Cao [31] for the nonsingular generalized saddle point problems to the singular generalized saddle point problems.

2 Semi-Convergence of the LHSS and MLHSS Iteration Methods

Denote $\rho(A)$ as the spectral radius of a square matrix A, $\lambda_{\max}(W)$ and $\lambda_{\min}(W)$ are the maximum and the minimum eigenvalues of an Hermitian positive definite matrix W, respectively. I is the identity matrix with appropriate dimension. $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ are the Hermitian and the skew-Hermitian parts of A, respectively.

Before the semi-convergence of the LHSS and MLHSS iteration methods for singular systems (1) are given, we first review the LHSS and MLHSS iteration methods proposed in [31] and give a lemma for latter use.

Method 2.1 ([31] LHSS Iteration Method). Assume that $Q_2 \in \mathbb{C}^{n\times n}$ is an Hermitian positive definite matrix, for initial vectors $x_0 \in \mathbb{C}^m$ and $y_0 \in \mathbb{C}^n$, the sequence $\{x_k, y_k\}$ is defined for $k = 1, 2, \cdots$ by

$$\begin{cases} x_{k+1} = x_k + H^{-1}(f - (Ax_k + By_k)), \\ y_{k+1} = y_k + Q_2^{-1}(B^* x_{k+1} - g). \end{cases} \tag{4}$$

Method 2.2 ([31] MLHSS Iteration Method). Assume that $Q_1 \in \mathbb{C}^{m\times m}$ is an Hermitian positive semi-definite matrix and $Q_2 \in \mathbb{C}^{n\times n}$ is an Hermitian positive definite matrix, for initial vectors $x_0 \in \mathbb{C}^m$ and $y_0 \in \mathbb{C}^n$, the sequence $\{x_k, y_k\}$ is defined for $k = 1, 2, \cdots$ by

$$\begin{cases} x_{k+1} = x_k + (Q_1 + H)^{-1}(f - (Ax_k + By_k)), \\ y_{k+1} = y_k + Q_2^{-1}(B^* x_{k+1} - g). \end{cases} \tag{5}$$

In fact, the LHSS method is the special case of the MLHSS method, and the above two methods are special cases of the inexact Uzawa method [25] when A is Hermitian positive definite matrix. The generalized inexact Uzawa method is proposed in [28] when A is the Hermitian positive definite matrix, which is the generalization of the inexact Uzawa method [25]. For the non-Hermitian positive definite matrix A, the generalized inexact Uzawa method is proposed in [31] as follows:

Method 2.3 ([31] Generalized Inexact Uzawa Method). Assume that $Q_1 \in \mathbb{C}^{m\times m}$ is an Hermitian positive semi-definite matrix and $Q_2 \in \mathbb{C}^{n\times n}$ is an Hermitian positive definite matrix, for initial vectors $x_0 \in \mathbb{C}^m$ and $y_0 \in \mathbb{C}^n$, the sequence $\{x_k, y_k\}$ is defined for $k = 1, 2, \cdots$ by

$$\begin{cases} x_{k+1} = x_k + (Q_1 + H)^{-1}(f - (Ax_k + By_k)), \\ y_{k+1} = y_k + Q_2^{-1}((1-t)B^* x_{k+1} + tB^* x_k - g). \end{cases} \tag{6}$$

where $t \in \mathbb{R}$ is a relaxation factor.

In fact, considering the following matrix splitting:

$$\begin{pmatrix} A & B \\ -B^* & 0 \end{pmatrix} = \begin{pmatrix} Q_1 + H & 0 \\ (t-1)B^* & Q_2 \end{pmatrix} - \begin{pmatrix} Q_1 - S & -B \\ tB^* & Q_2 \end{pmatrix},$$

the above generalized inexact Uzawa method (6) can be regarded as the following iterative method for solving system (1)

$$\begin{pmatrix} Q_1 + H & 0 \\ (t-1)B^* & Q_2 \end{pmatrix} \begin{pmatrix} x_{k+1} \\ y_{k+1} \end{pmatrix} = \begin{pmatrix} Q_1 - S & -B \\ tB^* & Q_2 \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix} + \begin{pmatrix} f \\ -g \end{pmatrix}, \tag{7}$$

and the corresponding iteration matrix is

$$\begin{aligned} T_t &= \begin{pmatrix} Q_1 + H & 0 \\ (t-1)B^* & Q_2 \end{pmatrix}^{-1} \begin{pmatrix} Q_1 - S & -B \\ tB^* & Q_2 \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \\ &= \begin{pmatrix} (Q_1 + H)^{-1}(Q_1 - S) & -(Q_1 + H)^{-1}B \\ T_{21} & I_n - (1-t)Q_2^{-1}B^*(Q_1 + H)^{-1}B \end{pmatrix} \end{aligned} \tag{8}$$

with $T_{21} = Q_2^{-1}B^*(Q_1+H)^{-1}(Q_1-S) + tQ_2^{-1}B^*A$. Hence, the iteration matrices of LHSS method and MLHSS method allow the following descriptions

$$T_1 = \begin{pmatrix} -H^{-1}S & -H^{-1}B \\ -Q_2^{-1}B^*H^{-1}S & I_n - Q_2^{-1}B^*H^{-1}B \end{pmatrix} \tag{9}$$

and

$$T_2 = \begin{pmatrix} (Q_1+H)^{-1}(Q_1-S) & -(Q_1+H)^{-1}B \\ Q_2^{-1}B^*(Q_1+H)^{-1}(Q_1-S) & I_n - Q_2^{-1}B^*(Q_1+H)^{-1}B \end{pmatrix}, \tag{10}$$

respectively.

For the following 2×2 partitioned matrix, its semi-convergence is described by the following lemma, see [24, 30].

LEMMA 2.1 ([24, 30]). Let $R \in \mathbb{C}^{l\times l}$ with positive integers l. Then the partitioned matrix

$$T = \begin{pmatrix} R & 0 \\ L & I \end{pmatrix}$$

is semi-convergent if and only if either of the following conditions holds true:

(1) $L = 0$ and R is semi-convergent;

(2) $\rho(R) < 1$.

It is worth pointing out that the convergence condition of Theorem 2.2 in [31] is a sufficient and necessary condition. Using the analogous proof in [24] and the same method in [31] for nonsingular systems (1), we will prove some analogous results on the semi-convergence for singular systems (1).

THEOREM 2.1. Assume that $r < n$, A is a non-Hermitian matrix with the positive-definite Hermitian part $H = \frac{1}{2}(A+A^*)$ and the skew-Hermitian part $S = \frac{1}{2}(A-A^*)$, i is the imaginary unit. Suppose that $[u^*, v^*]^*$ is an eigenvector according to an eigenvalue ($\neq 1$) of the iteration matrix T_2 of MLHSS method. Denote by

$$a = \frac{u^*Hu}{u^*u},\ -b = \frac{u^*i\cdot Su}{u^*u},\ c = \frac{u^*BQ_2^{-1}B^*u}{u^*u},\ d = \frac{u^*Q_1u}{u^*u}.$$

Then the MLHSS iteration method (5) is semi-convergent to a solution x of the singular systems (1) if and only if $a,\ b,\ c$ and d satisfy the following condition:

$$c < \frac{2a^3 + 4a^2d - 2ab^2}{a^2+b^2}. \tag{11}$$

PROOF. Let $B = U(B_r, 0)V^*$ be the singular value decomposition of B, $B_r = (\Sigma_r, 0)^T \in \mathbb{R}^{m\times r}$ with $\Sigma_r = \mathrm{diag}(\sigma_1, \sigma_2, ..., \sigma_r)$, $U,\ V$ are unitary matrices. Then

$$P = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$$

is an $(m+n)$-by-$(m+n)$ unitary matrix. Define $\widehat{T_2} = P^*T_2P$, the eigenvectors of $\widehat{T_2}$ are $[\hat{u}^*, \hat{v}^*]^*$, then the matrix $\widehat{T_2}$ has the same eigenvalues with matrix T_2, and

$[\hat{u}^*, \hat{v}^*]^* = P^*[u^*, v^*]^*$. Hence, we only need to demonstrate the semi-convergence of the matrix $\widehat{T_2}$.

Let $\widehat{A} = U^*AU$, $\widehat{H} = U^*HU$, $\widehat{S} = U^*SU$, $\widehat{Q_1} = U^*Q_1U$, $\widehat{B} = U^*BV$ and $\widehat{Q_2} = V^*Q_2V$. Then it holds that $\widehat{B} = (B_r, 0)$ and

$$\widehat{Q_2}^{-1} = \begin{pmatrix} V_1^*Q_2^{-1}V_1 & V_1^*Q_2^{-1}V_2 \\ V_2^*Q_2^{-1}V_1 & V_2^*Q_2^{-1}V_2 \end{pmatrix}$$

with appropriate partitioned matrix $V = (V_1, V_2)$. By simple computation, we have

$$\widehat{T_2} = \begin{pmatrix} \widehat{R} & 0 \\ \widehat{L} & I_{n-r} \end{pmatrix}$$

where

$$\widehat{R} = \begin{pmatrix} (\widehat{Q_1}+\widehat{H})^{-1}(\widehat{Q_1}-\widehat{S}) & -(\widehat{Q_1}+\widehat{H})^{-1}B_r \\ V_1^*Q_2^{-1}V_1B_r^*(\widehat{Q_1}+\widehat{H})^{-1}(\widehat{Q_1}-\widehat{S}) & I_r - V_1^*Q_2^{-1}V_1B_r^*(\widehat{Q_1}+\widehat{H})^{-1}B_r \end{pmatrix}$$

and

$$\widehat{L} = \left(V_2^*Q_2^{-1}V_1B_r^*(\widehat{Q_1}+\widehat{H})^{-1}(\widehat{Q_1}-\widehat{S}),\ -V_2^*Q_2^{-1}V_1B_r^*(\widehat{Q_1}+\widehat{H})^{-1}B_r\right).$$

As $\widehat{L} \neq 0$, from Lemma 2.1 we know that the matrix $\widehat{T}$ is semi-convergent if and only if $\rho(\widehat{R}) < 1$.

When the MLHSS method (5) applied to solve the following nonsingular generalized saddle point problems

$$\begin{pmatrix} \widehat{A} & B_r \\ B_r^* & 0 \end{pmatrix}\begin{pmatrix} \widehat{x} \\ \widehat{y} \end{pmatrix} = \begin{pmatrix} \widehat{f} \\ \widehat{g} \end{pmatrix} \tag{12}$$

with the preconditioning matrix $\widehat{Q_1}$ and $Q = (V_1^*Q_2^{-1}V_1)^{-1}$, and vectors $\widehat{y},\ \widehat{g} \in \mathbb{R}^r$, then the iterative matrix of the MLHSS method is $\widehat{R}$. From Theorem 2.2 of [31], we know that $\rho(\widehat{R}) < 1$ if and only if

$$\hat{c} < \frac{2\hat{a}^3 + 4\hat{a}^2\hat{d} - 2\hat{a}\hat{b}^2}{\hat{a}^2 + \hat{b}^2}, \tag{13}$$

with

$$\hat{a} = \frac{\hat{u}^*\widehat{H}\hat{u}}{\hat{u}^*\hat{u}},\ -\hat{b} = \frac{\hat{u}^*i\cdot\widehat{S}\hat{u}}{\hat{u}^*\hat{u}},\ \hat{c} = \frac{\hat{u}^*\widehat{B}_rQ^{-1}\widehat{B}_r^*\hat{u}}{\hat{u}^*\hat{u}},\ \hat{d} = \frac{\hat{u}^*\widehat{Q_1}\hat{u}}{\hat{u}^*\hat{u}}.$$

Note that the condition (13) is equivalent to the condition (11). By the above analysis, the proof is completed.

REMARK 2.1. In fact, Theorem 2.1 can be regarded as an extension of Theorem 2.2 in [31].

When $Q_1 = 0$, the MLHSS iteration method becomes the LHSS iteration method. Hence, the following Theorem gives a description on the semi-convergence of the LHSS method.

THEOREM 2.2. Assume that $r < n$, A is a non-Hermitian matrix with the positive-definite Hermitian part $H = \frac{1}{2}(A+A^*)$ and the skew-Hermitian part $S = \frac{1}{2}(A-A^*)$, i is the imaginary unit. Suppose that $[u^*, v^*]^*$ is an eigenvector according to an eigenvalue $(\neq 1)$ of the iteration matrix T_1 of LHSS method. Denote by

$$a = \frac{u^* H u}{u^* u},\ -b = \frac{u^* i \cdot S u}{u^* u},\ c = \frac{u^* B Q_2^{-1} B^* u}{u^* u}.$$

Then the LHSS iteration method (4) is semi-convergent to a solution x of the singular systems (1) if and only if $a,\ b$ and c satisfy the following condition:

$$c < \frac{2a(a^2-b^2)}{a^2+b^2}. \tag{14}$$

For the real case, there are better results about LHSS method (4) and MLHSS method (5), which are summarized in the following corollaries.

COROLLARY 2.1. Assume that $r < n$, A is a non-symmetric matrix with the positive-definite symmetric part $H = \frac{1}{2}(A + A^T)$ and the skew-symmetric part $S = \frac{1}{2}(A - A^T)$. Let Q_2 be symmetric positive definite matrix. Then the LHSS iteration method (4) is semi-convergent to a solution x of the singular systems (1) if and only if $2H - BQ_2^{-1}B^T$ is positive definite.

COROLLARY 2.2. Under the assumptions of Corollary 2.1, then the LHSS iteration method is semi-convergent if $\lambda_{\max}(BQ_2^{-1}B^T) < 2\lambda_{\min}(H)$.

COROLLARY 2.3. Under the assumptions of Corollary 2.1, if $Q_2 = \frac{1}{\delta}I$, then the LHSS iteration method is semi-convergent when $0 < \delta < \frac{2\lambda_{\min}(H)}{\lambda_{\max}(B^TB)}$.

COROLLARY 2.4. Assume that $r < n$, A is a non-symmetric matrix with the positive-definite symmetric part $H = \frac{1}{2}(A + A^T)$ and the skew-symmetric part $S = \frac{1}{2}(A - A^T)$. Let Q_1 be symmetric positive semi-definite and Q_2 be symmetric positive definite. Then the MLHSS iteration method (5) is semi-convergent to a solution x of the singular systems (1) if and only if $2H + 4Q_1 - BQ_2^{-1}B^T$ is positive definite.

COROLLARY 2.5. Under the assumptions of Corollary 2.4, then the MLHSS iteration method is semi-convergent if $\lambda_{\max}(BQ_2^{-1}B^T) < 2\lambda_{\min}(H) + 4\lambda_{\min}(Q_1)$.

COROLLARY 2.6. Under the assumptions of Corollary 2.4, if $Q_1 = \alpha I$, $Q_2 = \frac{1}{\delta}I$, then the MLHSS iteration method is semi-convergent when $0 < \delta < \frac{2\lambda_{\min}(H)+4\alpha}{\lambda_{\max}(B^TB)}$.

When the generalized inexact Uzawa method (6) is applied to solve the real singular systems (1), the following Theorem describes the semi-convergence of the generalized inexact Uzawa method (6).

THEOREM 2.3. Assume that $r < n$, A is a non-symmetric matrix with the positive-definite symmetric part $H = \frac{1}{2}(A + A^T)$ and the skew-symmetric part $S = \frac{1}{2}(A - A^T)$. Suppose that $[u^T, v^T]^T$ is an eigenvector according to an eigenvalue of the iteration matrix T_t of generalized inexact Uzawa method. Denote by

$$a = \frac{u^T H u}{u^T u},\ c = \frac{u^T B Q_2^{-1} B^T u}{u^T u},\ d = \frac{u^T Q_1 u}{u^T u}.$$

Then the generalized inexact Uzawa method (6) is semi-convergent to a solution x of the singular systems (1) if and only if a, c, d and t satisfy the following condition:

$$a - tc > 0 \ \text{ and } \ 2a + 4d + (2t-1)c > 0. \tag{15}$$

COROLLARY 2.7. Under the assumptions of Theorem 2.3, the generalized inexact Uzawa method (6) is semi-convergent if and only if

$$H - tBQ_2^{-1}B^T \ \text{ and } \ 2H + 4Q_1 + (2t-1)BQ_2^{-1}B^T$$

are positive definite.

3 Conclusion

In this paper, we further investigate the LHSS and MLHSS iteration methods presented in [31] for solving singular linear systems (1). When A is non-Hermitian positive definite and the Hermitian part of A is dominant, the semi-convergence conditions are proposed, which generalize some results of Jiang and Cao [31] for the nonsingular generalized saddle point systems to the generalized singular saddle point systems.

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