Applied Mathematics E-Notes, 11(2011), 1-11 ©
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ISSN 1607-2510

Face Labeling Of Maximal Planar Graphs*

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Received 21 September 2009

Abstract

The labeling problem considered in this paper is called face-labeling of the maximal planar or triangular planar graphs in connection with the notion of the *consistency*. Several triangular planar and maximal planar graphs such as the wheels, the fans etc. are considered.

1 Introduction

Graph labeling problems are quite different than the graph coloring problems [5] but the author has an unproved claim that every coloring problem can be expressed as an labeling problem.

In this paper, we will consider only consistency of the face labelings (f-labeling in short) and give several results on special triangular planar graphs such as the fans, wheels, triangular chains as well as maximal planar graphs. We will also consider the inconsistency of f-labeling for these graphs which eventually enable us to give an face-labeling algorithm for any maximal planar graph. According to the author this observation should be counted extraordinarily coincidental, partly because of his long struggle with the notorious graceful tree conjecture [15],[16]. In the next section, after giving some basic results for simple triangular graphs, in particular we investigate in detail the f-labeling of the fans, the wheels and triangular chains. The importance of these graphical structures comes from the fact that, they can be considered as the basic building blocks of the maximal planar graphs after the triangle. In Section 3 we consider a decomposition algorithm for the maximal planar graphs into fans and wheels. The last section deals with the consistency of f-labeling of maximal planar graphs based on the decompositions given in the previous section.

2 Basic Results

Let $\triangle$ denote a triangle (a cycle of length three). Let $1, 2, 3, 4$ be the possible labels corresponding to the different colors. Then by considering all possible vertex labelings

*Mathematics Subject Classifications: 05C10, 05C15, 05C78.

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of a triangle $\triangle$ we get the following table:

Face Labels	$x_6 = 6$	$x_7 = 7$	$x_8 = 8$	$x_9 = 9$
Consistent labels	$\{1,2,3\}\Delta$	$(1,2,4\}\Delta$	$\{1,3,4\}\Delta$	$(2,3,4\}\Delta$
Inconsistent labels	$\{1,1,4\}\Delta$	$\{2,2,3\}\Delta$	$\{2,2,4\}\Delta$	$\{3,3,3\}\Delta$
Inconsistent labels	$\{2,2,2\}\Delta$	$\{1,3,3\}\Delta$	$\{2,3,3\}\Delta$	$\{1,4,4\}\Delta$

The first row in the table corresponds to the proper coloring of the triangle while the other rows are not. In notation, we will denote face labels by x_i, x_j, x_k, x_l and drop the symbols f and $\triangle$ whenever the context permits and denote the induced vertex labels by y_i, y_j, y_k, y_l, where indices of x's are the actual face labels $6, 7, 8, 9$ and the indices of y's are the actual vertex labels $1, 2, 3, 4$.

DEFINITION 1. A labelling $f : F \to \{6, 7, 8, 9\}_f$ of the faces of a maximal planar graph G is called *consistent* if for every triangle $\triangle$ of G there exists induced vertex labelings in the form $x_6 = \{1, 2, 3\}_v, x_7 = \{1, 2, 4\}_v, x_8 = \{1, 3, 4\}_v, x_9 = \{2, 3, 4\}_v$. That is, if y_1, y_2, y_3 are the vertices of a triangle $\triangle$ then $f\{y_1, y_2, y_3\}_\triangle = f_v(y_1) + f_v(y_2) + f_v(y_3) = 6, 7, 8$,or 9, where $f(x_i) = f\{y_{1,} y_2, y_3\}$ is the face label of the triangle $\triangle$. Otherwise, the face labeling f is called *inconsistent*.

With the above notations for consistent face labelings of a triangle we can write:

$\{x_6\} \cap \{x_7\} = \{y_1, y_2\}$
$\{x_6\} \cap \{x_8\} = \{y_1, y_3\}$
$\{x_6\} \cap \{x_9\} = \{y_2, y_3\}$
$\{x_7\} \cap \{x_8\} = \{y_1, y_4\}$
$\{x_7\} \cap \{x_9\} = \{y_2, y_4\}$
$\{x_8\} \cap \{x_9\} = \{y_3, y_4\}$
$\{x_6\} \cap \{x_7\} \cap \{x_8\} = \{y_1\}$
$\{x_6\} \cap \{x_7\} \cap \{x_9\} = \{y_2\}$
$\{x_6\} \cap \{x_8\} \cap \{x_9\} = \{y_3\}$
$\{x_7\} \cap \{x_8\} \cap \{x_9\} = \{y_4\}$ and
$\{x_6\} \cap \{x_7\} \cap \{x_8\} \cap \{x_9\} = \emptyset$.

The last term is important since it indicates that no set of four triangles with a common vertex v has a consistent the face labeling if all possible face labels x_6, x_7, x_8, x_9 have been assigned to the triangles. In Figure 1, we illustrate several consistent and in Figure 2 inconsistent face labelings of simple triangular graphs.

We begin with a simple lemma.

LEMMA 1. Let G_1 and G_2 be any two triangular planar graphs with consistent f-labelings f_1 and f_2. Then $G_1 \cup G_2$ has a consistent labeling $f_1 \cup f_2$.

PROOF. Assume that, the union of the two triangular graphs G_1 and G_2 are to be joined over a common edge e. But for any face labelings of two triangles we have $x_i \cap x_j = \{y_p, y_q\}, i, j = 6, 7, 8, 9$ and $p \neq q, p, q = 1, 2, 3, 4$. Thus the union of two consistent triangular planar graphs in this way results in another consistent triangular graph with the face labeling $f = f_1 \cup f_2$.

It can be easily verified that if the common edges between the two overlapped triangular planar graphs is more than a single edge then the union of consistent face labelings may not result in another consistent face labeling.

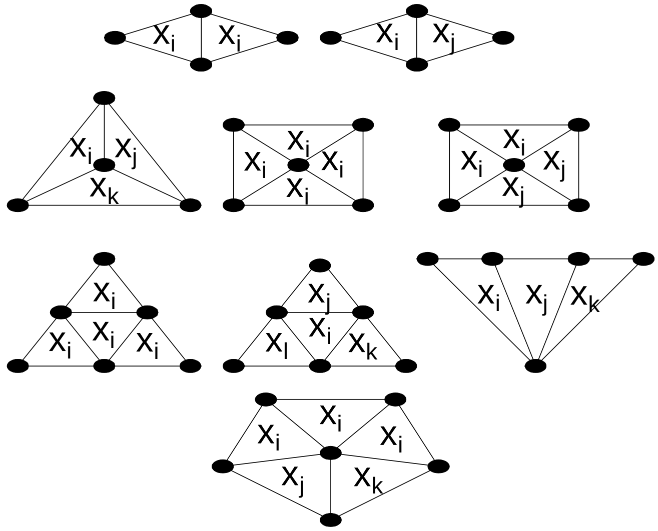

Figure 1: Consistent Face Labelings

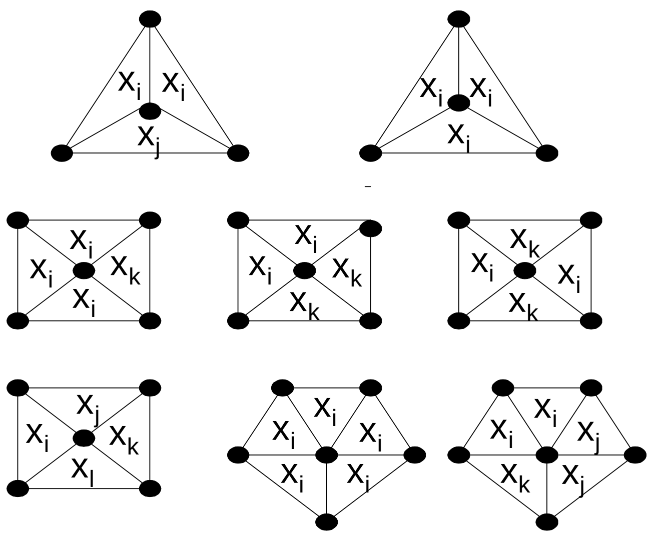

Figure 2: Inconsistent Face Labelings

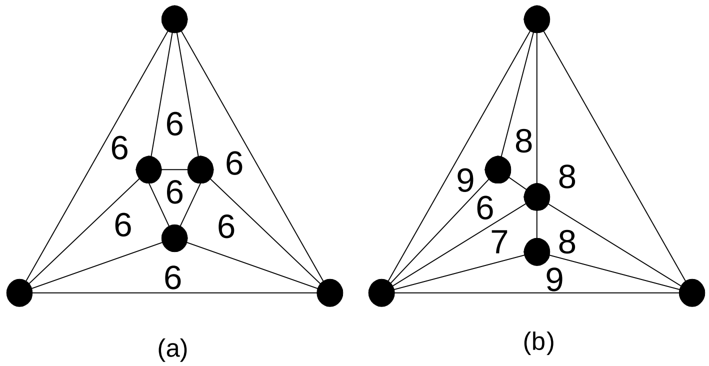

Figure 3: Pyramid Graphs

LEMMA 2. Any face labeling of the pyramid triangular graph with four triangles shown in Figure 3(a) is consistent.

PROOF. Let t_i be the inner triangle and t_a, t_b, t_c be the triangle neighbor t_i of the pyramid graph. Let us label t_i with $x_i, i \in \{6, 7, 8, 9\}$. The corner vertices not incident at t_i are free and any face label can be assigned to the triangles t_a, t_b and t_c due to the fact that $x_i \cap x_j$ is consistent for any face labeling. Therefore the overall pyramid graph is consistent for any face labeling.

LEMMA 3. Consider any planar drawing of K_4. Then the only consistent f-labelings of K_4 for which the facial faces are labelled by $x_i, x_j, x_k, i \neq j \neq k, i, j, k \in \{6, 7, 8, 9\}$ and the outerface label is $x_l = y_a + y_b + y_c$, where y_a, y_b, y_c are the induced vertex labels of the outerboundary face.

PROOF. For any planar drawing of K_4 there are exactly four faces including the outerface. Since each induced vertex label y_i ,$1 \leq i \leq 4$ used for a vertex exactly once and any two triangles have exactly one edge in common there exists only one consistent f-labeling for which all face labels are different.

LEMMA 4. A planar triangular graph G is consistent under single face label iff it is $3-$colorable.

PROOF. If G is 3-colorable then for each triangle we have the same induced vertex labels $y_i, y_j, y_k, i \neq j \neq k$. Now assume that G is not 3-colorable. By the four color theorem [1] there must be two triangles for which two vertex labels are different. Hence G is not consistent under single face label when it is four colorable.

We can ask whether any triangular graph can be labeled with less than four distinct face labels. The face labeling of the maximal planar graph shown in Figure 3(b) requires all four face labels to be used while for the maximal planar graph shown in Figure 3(a) a single face label (6 in the figure) is enough for a consistent face labeling.

The fan F_n $(n \geq 2)$ is obtained by joining all vertices of the path P_n to a further vertex called the center, and contains $n + 1$ vertices and $2n - 1$ edges. The wheel W_n $(n \geq 3)$ is obtained by joining all vertices of the cycle C_n to the center, and contains

$n+1$ vertices and $2n$ edges. The triangular path T_p is consisted of k triangles in the form of a path in which except the first and last triangles all other triangles pair wise share common edges. Similarly the triangular chain T_c can be defined by allowing the fans as subgraphs in the triangular path. In Figures 1 and 2 we illustrate consistent and inconsistent face labelings of the several simple triangular graphs.

THEOREM 1. The fan F_n (and the wheel W_n) has an consistent f-labeling iff it does not contain a subsequences of the face labels in the forms

(a) ($i \neq j$), $...x_i x_j x_j ... x_j x_j x_i ...$, where the number of x_j's is *odd*

(b) $(i \neq j \neq k), ...x_i x_j x_j ... x_j x_j x_k ...$, where the number of x_j's is *even* and

(c) the complete set $x_i, x_j, x_k, x_l, (i \neq j \neq k \neq l)$ is not assigned to the faces of the F_n.

PROOF. The proof is only given for the fans. For wheels similar lines follow.

Necessity of (c) is clear if we consider $x_i \cap x_j \cap x_k \cap x_l = \emptyset$. That is, let v be the vertex of F_n such that $F_n - v$ is a path. Assume that face labels x_i, x_j, x_k, x_l have been assigned to the some faces of F_n and the labeling is consistent. W.l.o.g. let $x_i = x_6 = \{y_1, y_2, y_3\}, x_j = x_7 = \{y_1, y_2, y_4\}, x_k = x_8 = \{y_1, y_3, y_4\}$ and $x_l = x_9 = \{y_2, y_3, y_4\}$. Since v is common for these triangles, for consistent face labeling mutual intersections of x's should not be empty. Next consider the subsequence $x_i x_j ... x_j x_i$ as a partial face labeling of F_n. If we start with the face label of x_i then the induced vertex label of the vertex v is determined together with the vertex label, say $y_i, 1 \leq i \leq 4$, on the path that adjacent to the first triangle, say from the left, x_j. Hence the unlabeled vertex of that triangle is uniquely determined. In order to complete partial face labeling, the sequence consistently, the rightmost triangle must have the induced vertex label y_i. But this is only possible if the number of triangles between the first triangle x_i and the last triangle x_i is even. Now consider any subsequence of face labels of F_n in the form $...x_i x_j x_j ... x_j x_k ..., (i \neq j \neq k)$ with even number of x_i face labels. Again for the consistency the common vertex v must receive the induced vertex label $x_i \cap x_j \cap x_k = y_p, p \in \{1, 2, 3, 4\}$.The remaining three vertex labels, say y_q, y_r, y_s, are pairwise elements of x_i, x_j and x_k. That is we may write $x_i = \{y_p\} \cup \{y_q, y_r\}, x_j = \{y_p\} \cup \{y_q, y_s\}, x_k = \{y_p\} \cup \{y_r, y_s\}$. On the other hand we have assumed that the path $F_n - v$ has even number of edges. In order to have consistent face labeling we must have the following vertex labels on the vertices of this path: $...y_r, y_q, y_s, y_q, ..., y_q, y_s, y_r ....$ But this is only possible if the number y_q, y_s pairs is odd.

For sufficiency, we will show that any other face labeling results in consistent induced vertex labeling.

Let v be the central vertex and let $P(x, y)$ be the path connecting degree two vertices x and y of F_n. All other vertices of $P(x, y)$ are of degree three.

Let $S = \{x_{i_1}, x_{i_2}, ..., x_{i_k}\}$ be the sequence of face labels of F_n satisfying the conditions (a),(b) and (c). W.l.o.g. assume also that S is only consisted of x_6, x_7 and x_8's (condition (c)). Hence vertex label of v is $f(v) = x_6 \cap x_7 \cap x_8 = 1$. Let us first consider the first two face labels from the left side in S.

$\{x_{i_1}, x_{i_2}\} = \{\{x_6, x_6\}, \{x_6, x_7\}, \{x_6, x_8\}, \{x_7, x_6\}, \{x_7, x_7\}, \{x_7, x_8\}, \{x_8, x_8\},$
$\{x_8, x_7\}, \{x_8, x_6\}\}$

The induced vertex label sets for each face label pairs, respectively, are:

$\{x_6, x_6\} \Leftrightarrow \{y_2, y_3, y_2\}or\{y_3, y_2, y_3\}, \{x_6, x_7\} \Leftrightarrow \{y_3, y_2, y_4\}, \{x_6, x_8\} \Leftrightarrow \{y_2, y_3, y_4\},$
$\{x_7, x_6\} \Leftrightarrow \{y_4, y_2, y_3\}, \{x_7, x_7\} \Leftrightarrow \{y_2, y_4, y_2\}$ or $\{y_4, y_2, y_4\}, \{x_7, x_8\} \Leftrightarrow \{y_2, y_4, y_3\}$
$\{x_8, x_6\} \Leftrightarrow \{y_4, y_3, y_2\}, \{x_8, x_7\} \Leftrightarrow \{y_3, y_4, y_2\}, \{x_8, x_8\} \Leftrightarrow \{y_4, y_3, y_4\}$ or $\{y_3, y_4, y_3\}$.

Now we can augment the above pairs of face labels to the triples by adding new face label $x_{i_3} \in \{x_6, x_7, x_8\}$ e.g., if $x_{i_3} = x_6$ then except the second and the third (condition (a)) all other pairs augmented consistently into the triple face labels. Clearly we can obtain any consistent face labeling of F_n in this way.

LEMMA 5. Let W_n be the wheel graph with $(n + 1)$ vertices. Then for any consistent f-labeling the maximum number of faces labeled by x_i is n if n =even and $(n - 2)$ otherwise.

PROOF. If n is even we can assign any fixed face label $x_i \in \{6, 7, 8, 9\}$ to all faces of the W_n. Assume that, we have selected $x_i = 6$. Then the induced vertex label for the central vertex of W_n can be any label from the set $\{1, 2, 3\}$. The other two vertex labels, say $2, 3$, if 1 is assigned to the center, are alternatingly assigned to the spoke vertices in which no two adjacent labels are same. If n is odd we can still do the same thing as above except for the last two faces. We have to use face labels other than 6. Otherwise inconsistency have occurred. If we label these faces with different labels from $\{7, 8, 9\}$'s then by Theorem 1 we create an inconsistent subsequence in the form $x_i x_j ... x_j x_k$ with $i \neq j \neq k$, and odd number of x_i's . Therefore we use any fixed label from $\{7, 8, 9\}$ for the remaining last two unlabeled faces of W_n. Here if the three distinct face labels x_i, x_j, x_k are used in W_n then the central vertex of W_n must be labeled with $x_i \cap x_j \cap x_k$.

3 Face Labeling of Triangular Graphs

In order to show the difficulty of finding consistent f-labeling of a triangular graph, consider the triangular graph consisted of three complete subgraphs with four vertices (K_4) which cyclically pairwise sharing common edges shown in Figure 4 together with a consistent face labeling. The subgraph induces by the faces labeled x_j, x_l, x_k is a wheel W_7 (shown in bold lines in the figure). If we label the faces of W_7 with all x_i's we would obtain another consistent face labeling of W_7 but cannot complete the rest of the graph consistently. Another difficulty is the possibility of creating inconsistent face label on a inner face which has not contained in any wheel or fan subgraphs. In order to detect situations like this beforehand, we need an algorithm to list triangles which completely depends on its boundary subgraphs and give priority to these triangles in the face labeling algorithm. Let us denote by $N(v)$ the neighborhood of the vertex (excluding the vertex v).

LEMMA 6. If G is a maximal planar graph then the neighborhood $N(v)$ of any vertex $v \in V(G)$ induces a wheel subgraph.

PROOF. Let us assume that there exists an vertex $v \in V(G)$ for which $N(v)$ is not induce an spanning cycle C of length $|N(v)|$. Let $k = |N(v)| - 1$ and v_{k+1} is not contained in the spanning cycle C. The vertex v_{k+1} must adjacent to another vertex, say $v_1 \in N(v)$, other than v. Now we can embed the planar graph G so that the cycle

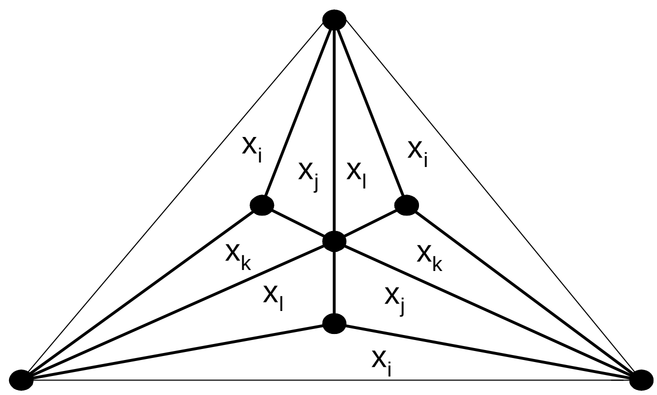

Figure 4: Consistent Labeling of a Maximal Planar Graph

$\{v_{k+1}, v, v_1, v_k\}$ without a chord becomes boundary of the outerface which contradicts to the maximality of G. Therefore $(v_{k+1}, v_k) \in E(G)$ which shows $|N(v)| = k$.

Base on the existing proofs of the four color theorem we can safely state [4],[7]:

THEOREM 2. Every maximal planar graph has a consistent f-labeling.

In order to attempt to give a different proof to the above theorem, we need firstly a decomposition and secondly an f-labeling algorithms that assures the consistency of face labeling.

DEFINITION 2. Let $D(G)$ be a decomposition of a maximal planar graph G into wheel subgraphs $W_{i_1}, W_{i_2}, ..., W_{i_k}, i_j > 3, j = 1, 2, ..., k$ and into fan subgraphs $F_{i1}, F_{i_2}, ..., F_{i_k}, i_j > 4, j = 1, 2, ..., k$. A triangle (face) of G is called a *black hole* if its edges contained in the three different wheel or fan subgraphs in $D(G)$.

In Fig.5 we illustrate the notion of black holes. The cycles $h_1 = \{c, d, g\}, h_2 = \{a, d, e\}, h_3 = \{e, g, l\}, h_4 = \{a, c, l\}$ are the black holes of the maximal planar graph since they have common edges with the wheel $W_5(K_4)$ subgraphs induce by the vertex sets $\{a, b, c, d\}, \{d, f, e, g\}, \{c, g, h, l\}$.

Note that, the last three black holes have common edges with the exterior face of the graph, i.e., $\{a, e, l\}$. The following is our *naive algorithm* for the consistent f-labeling of a maximal planar graph G. Assume that, G is drawn on the plane without crossing any edges.

- **Step 1**. *Find the decomposition* $D(G)$ *of* G *into the wheel (and fan) subgraphs.*
- **Step 2**. *Based on the decomposition* $D(G)$ *list all black holes* $h_1, h_2, ..., h_k$.

By using Theorem 5 and avoiding the traps around the holes, label all other faces other than $h_1, h_2, ..., h_k$.

The trap occurs at the hole h_i *if the face labeled adjacent* h_i *creates inconsistent face label on the hole.*

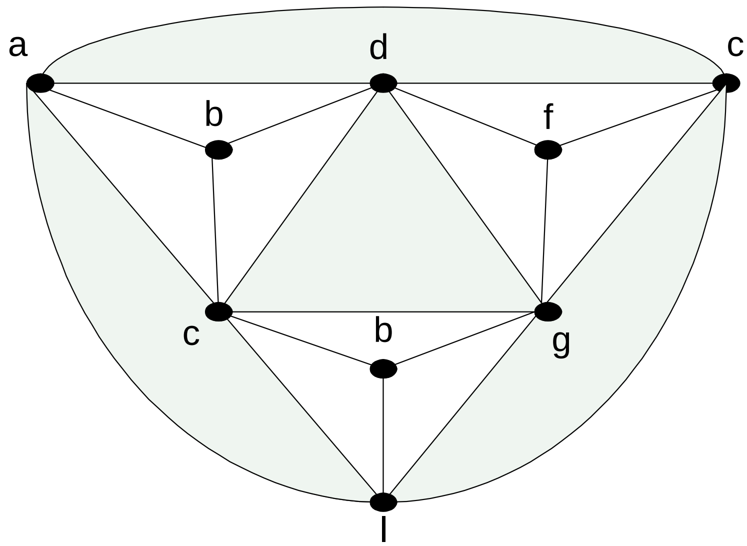

Figure 5: Black Holes of $D(G)$

4 Conclusion

Saaty has listed some 29 equivalent formulations of the famous four color theorem [6],[14] . After so many formulations, it is quite strange that the existing solutions to this problem is almost unique in the sense of methodology, i.e., mainly on the nonexistence of an counterexample and in the technique used in the proof of the four color theorem, e.g., configurations and reducibility, discharging, and a coloring algorithm [1]-[4],[7],[8] (e.g., we may refer the historical developments for the proofs in [9],[10]-[13]). Even the solvers of this problem implicitly disclose their feelings about the dissatisfaction of forcibly the extensive use of a computer [8]. Note that this is not due to the use of an quadric or quadratic coloring algorithms but it is the use of the computer for simply checking or verifications of the huge possibilities.

For example, let $N = \{1, 2, 3, 4\}$ be the set of possible vertex labels. Consider the vertex labeling $f : V \to \{1, 2, 3, 4\}$ of an planar graph G such that then the induced edge labels calculated as $f(x) - f(y)$ we require no *zero* edge label i.e., $f(x) \neq f(y)$, for all $(x, y) \in E(G)$. Clearly, this is exactly the same thing as the 4CP formulation, we simply used integers $1, 2, 3, 4$ instead of the colors, say blue, red, green and yellow. It is also clear that this kind of graph labeling would not give further possibilities to investigate the four-color problem since the knowledge remain local. That is the rule of proper coloring of two adjacent vertices remain same by assigning to different vertex labels. The graph theoretical version of the 4CT:

THEOREM 3. The vertices of every maximal planar graph with at least four vertices can be colored with at most four colors.

The key word in the above statement is the word “maximal” that enable us to give

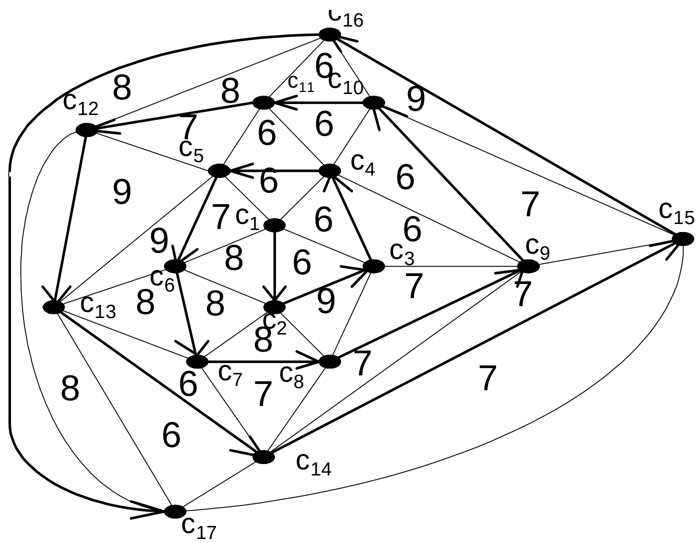

Figure 6: Consistent Labeling of a Triangulation [14]

quite different labeling equivalent of the four color problem. Maximal planar graphs are such that addition of any other edges results in an nonplanar graph. The number of the edges in any maximal planar graph with n vertices is $3n - 6$. Furthermore the (facial) faces (including the infinite outerface) are all triangles e.g., cycles of length three, where a facial triangle is a triangle in G which is the boundary of some face of G. Another motivation to the face labeling considered is due to the simple proof of the following:

THEOREM 4. The faces of every maximal planar graph are three colorable except the complete graph on four vertices.

The proof of this theorem is that, the dual of a maximal planar graph is 3-regular. By Brooks' Theorem, it is 3-colorable if it is not a clique[1]. Again if we assume the truth value of the above theorem *a priori* [1]-[4],[7],[8],[14] there is nothing to give and nothing to prove, since all maximal planar graphs have at least one consistent face labeling. However, our hope, is to open an avenue for a possible proof of the well-known four color theorem without using a computer. We have not yet attempted to the algorithmic solution of the four color problem but the formulation of the problem in terms of face labeling enable us to learn global property of the coloring of maximal planar graphs. That is, it is equally important to know under what conditions the subgraphs of the maximal planar graph would not have consistent face labeling. We conjectured, that the notion of face labeling is sufficient to give a proof of theorem (4CT) without using a computer.

In Figure 6, we have selected the triangulation given by R. Thomas in [14] together with the f-labeling. The f-labeling is based on consistent face labeling of every pairs

[1] The proof has been provided by Douglas West in response to the question via the GRAPHNET.

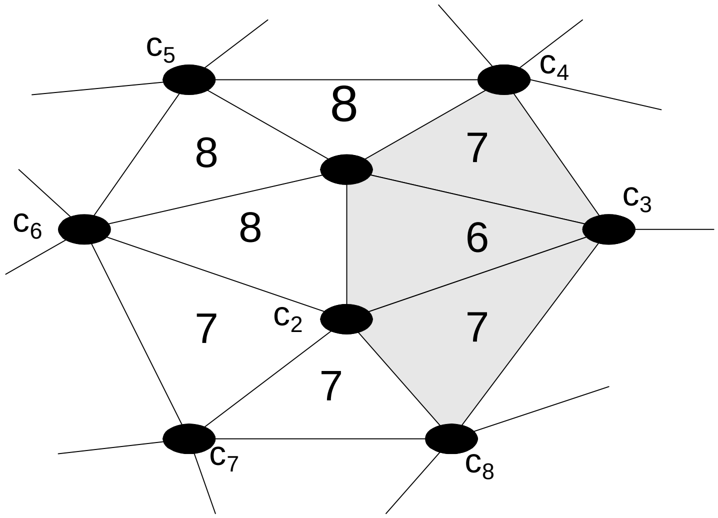

Figure 7: Detecting Inconsistent Subsequence Face Labels $7, 6, 7$

of all wheel subgraphs by considering the result of Theorem 1. Note that in the triangulation of Figure 6 there is no black holes. Certainly, the order of the selection of the wheel subgraphs is another aspect of the face labeling algorithm (ordering of the wheels have been shown bold edges with arrows indicating the center of the respective wheel subgraph of the maximal planar graph). We choose the inner and smallest sized wheel first and continue next to the face labeling of the neighbor wheel subgraphs based on the direction of directed edges. Actually the directed edges in the graph constitute topologically in the shape of a *spiral* directed path. In this way we guarantee that the unlabeled wheel subgraph *en route* will have adjacencies with the minimum number of the previously labeled wheels. However care must be given at the current face labeling of the wheel subgraph. For example, if we consistently label the faces of the first and second wheel subgraphs induce by the vertices $W(c_1) = \{c_1, c_2, c_3, c_4, c_5, c_6\}$ and $W(c_2) = \{c_2, c_1, c_3, c_8, c_7, c_6\}$ as shown in Figure7 we would create inconsistent face labeling on the faces of the fan subgraph induces by the vertices c_1, c_2, c_8, c_3, c_4, i.e., the subsequence $7, 6, 7$ (see Theorem 1).

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