On The Concavity Of The First NLPC Transformation Of Unimodal Symmetric Random Variables

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Abstract

We study the concavity of the first NLPC transformation for symmetric unimodal distributions on bounded domains. We deduce a comparison principle based on the variances of the first NLPC and show a possible application in constructing goodness-of-fit tests.

1 Introduction

Let $X$ be an absolutely continuous random variable (r.v.) with zero mean, finite variance and density $f_X$ having support the closure $\overline{D}$ of an interval $D$ ($\Upsilon (D)$ will denote the set of these r.v.s). As introduced in [6], the first nonlinear principal component (NLPC) of $X$, if it exists, is the r.v. $\varphi_1 (X)$ where $\varphi_1$ is defined as

$$\varphi_1 = \arg \max_{u \in \hat{W}^{1,2}_X (\{0\})} \mathbb{E} \left[ u(X)^2 \right] \left( \mathbb{E} \left[ u'(X)^2 \right] \right)^{-1}. \quad (1)$$

Here $\hat{W}^{1,2}_X = \{ u \in \mathcal{L}^2_X : u' \in \mathcal{L}^2_X \}$ and $\mathcal{L}^2_X$ (resp. $\mathcal{L}^2_X^c$) is the separable Hilbert space of centered (resp. not necessarily centered), square integrable functions $u : D \to \mathbb{R}$. We will assume $(1/f_X) \in \mathcal{L}^1_{\text{loc}} (D)$, thus $\hat{W}^{1,2}_X$ is Hilbert too. By (1) $\varphi_1$ realizes the equality in the Poincaré inequality (see e.g. [2], [3], [4], [5], [7], [8]):

$$\exists C > 0 : \quad \text{Var} \left[ u(X) \right] \leq C \mathbb{E} \left[ (u'(X))^2 \right] \quad (2)$$

and the variance $\lambda_1$ of $\varphi_1 (X)$ coincides with the optimal Poincaré constant $C$. Some properties of $\varphi_1$ are collected in the following lemma (see [6]):

Lemma 1. Suppose $X \in \Upsilon (D)$ admits NLPCs and let $\varphi_1$ be the first NLPC transformation. The following conclusions hold:

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Concavity of NLPC Transformation

(i) if $f_X$ is even, then $\varphi_1$ is odd;
(ii) if $f_X \in C^1(D)$, then $\varphi_1$ is strictly monotone;
(iii) if $\varphi_1 \in C^2(D)$ then $f_X = g/ \int_D g$, where $g(x) = (\varphi_1'(x))^{-1} \exp \{-\xi_1 \int \varphi_1/\varphi_1'\}$ and $\xi_1 = 1/\lambda_1$.

Here and in the following $C^k(D)$ denotes as usual the set of $k \geq 0$ times continuously differentiable real functions defined on $D$.

Note how statement (iii) highlights the central role of $\varphi_1$ in characterizing the distribution of $X$, justifying the interest in deepening the knowledge of its properties.

We will assume, without loss of generality, $\xi \in (0,1)$. Differentiating in (3), we get

$$
\varphi_1''(x) = -\frac{f_X''(x)}{f_X(x)} \varphi_1'(x) - \xi_1 \varphi_1(x), \quad \forall x \in [0,1].
$$

2 Main Results

We recall that $\varphi_1$ is a weak solution, with $\xi = \xi_1 = \lambda_1^{-1}$, of the Sturm-Liouville problem:

$$
\begin{cases}
- (f_X u')' = \xi f_X u, & \text{in } D \\
\lim_{x \to a^+} u'(x) f_X(x) = \lim_{x \to b^-} u'(x) f_X(x) = 0
\end{cases}
$$

that is $\varphi_1 \in \dot{W}_X^{1,2}$ and $E[\varphi_1(X) h'(X)] = \xi E[\varphi_1(X) h(X)]$ for all $h \in \dot{W}_X^{1,2}$, whereas $\varphi_1$ is a strong solution of (3) if $f_X \varphi_1' \in C^0(D) \cap C^1(D)$, that is $\varphi_1$ satisfies (3) pointwise.

We will assume, without loss of generality, $D = (-1,1)$ and $X \in \mathcal{H}(D)$ such that:

(H1) it admits first NLPC $\varphi_1$;

(H2) its density $f_X \in C^0[-1,1] \cap C^1(-1,1)$ is symmetric and unimodal at 0, with $f_X' \leq 0$ on $(0,1)$.

For the sake of shortness we will denote by $\mathcal{H}(D)$ the set of such r.v.s.

PROPOSITION 1. Let $X \in \mathcal{H}(D)$ with $f_X \in C^2(-1,1)$ and assume

$$
A(x) := -\frac{d^2}{dx^2} \ln(f_X(x)) - \xi_1 \quad x \in [0,1]
$$

is such that (i) $A(0) < 0$; (ii) $A$ has at most one zero in $(0,1)$ in which it changes sign. Then $\varphi_1$ is concave in $[0,1]$.

PROOF. By (H1) and (H2) $\varphi_1$ is a strong solution of (3). Since $f_X \in C^1(-1,1)$ we obtain $\varphi_1 \in C^2(-1,1)$; moreover by $f_X \in C^2(-1,1)$ and (3) and it follows $\varphi_1 \in C^3(-1,1)$. Differentiating in (3), we get

$$
\varphi_1''(x) = -\frac{f_X''(x)}{f_X(x)} \varphi_1'(x) - \xi_1 \varphi_1(x), \quad \forall x \in [0,1].
$$
Since \( f'_X(0) = 0 \), \( \varphi_1(0) = 0 \) and \( \varphi'_1(0) > 0 \), we have \( \varphi''_1(0) = 0 \). Differentiating in (5) we obtain
\[
\varphi''_1(x) = \varphi'_1(x) A(x) - \varphi''_1(x) \frac{d}{dx}(\ln(f_X(x))) \quad \forall x \in [0, 1),
\] (6)
from which \( \varphi''_1(0) = \varphi'_1(0) A(0) < 0 \). Thus \( \varphi'_1(x) < 0 \) in a right neighborhood of 0 (recall that \( \varphi'_1 \in C^0(-1, 1) \)).

Assume first \( A(x) < 0 \) in \((0, 1)\). Since \( \varphi'_1(x) > 0 \) in \((-1, 1)\), if there exists \( x_1 \in (0, 1) \) such that \( \varphi'_1(x_1) = 0 \) from (6) it follows \( \varphi''_1(x_1) < 0 \), a contradiction; thus \( \varphi'_1(x) < 0 \) in \((0, 1)\) and we conclude.

Assume now that there exists (a unique) \( \overline{x} \in (0, 1) \) such that \( A(\overline{x}) = 0 \) and \( A(x) > 0 \) in \((\overline{x}, 1)\), hence
\[
\frac{d^2}{dx^2} \ln(f_X(x)) \leq -\xi_1 \quad \text{for all } x \in [\overline{x}, 1).
\] (7)

We show first that
\[
\limsup_{x \to 1^-} \varphi''_1(x) < 0.
\] (8)
If \( \limsup_{x \to 1^-} -f'_X(x)/f''_X(x) = c \in [0, +\infty) \), since \( \lim_{x \to 1^-} f_X(x) = 0, \) (8) easily follows from (5).

Suppose \( \limsup_{x \to 1^-} -f'_X(x)/f''_X(x) = +\infty \). Condition (7) assures that the function \( f'_X(x)/f_X(x) \) is strictly decreasing in \((\overline{x}, 1)\), hence it exists \( \lim_{x \to 1^-} f'_X(x)/f_X(x) = \alpha \) with \( \alpha \in (-\infty, 0) \). With some computations one deduces \( \lim_{x \to 1^-} f'_X(x)/f''_X(x) = 0 \). Then, we get
\[
\limsup_{x \to 1^-} \frac{-f'_X(x)}{f''_X(x)} \varphi'_1(x) = \limsup_{x \to 1^-} \frac{-\varphi'_1(x)f_X(x)}{f'_X(x)(f_X(x))^{-1}} \leq \limsup_{x \to 1^-} \frac{-\varphi'_1(x)f_X(x)}{(f'_X(x)(f''_X(x))^{-1})'}
\]
\[
\leq \limsup_{x \to 1^-} \xi_1 \varphi'_1(x) \left[ 1 + \frac{d^2}{dx^2} \ln(f_X(x)) \left( \frac{d}{dx} \ln(f_X(x)) \right)^{-2} \right]^{-1}
\]
\[
\leq \limsup_{x \to 1^-} \xi_1 \varphi'_1(x) \left[ 1 + \xi_1 \left( \frac{d}{dx} \ln(f_X(x)) \right)^{-2} \right]^{-1}
\]
\[
< \xi_1 \lim_{x \to 1^-} \varphi'_1(x),
\]
where again we use (7). By this, (8) follows.

Now suppose by contradiction that \( \varphi''_1 \) changes sign in \((0, 1)\) and let \( x_1, x_2 \in (0, 1) \), with \( x_1 < x_2 \), be its “first and last” zeroes, respectively. By (6) and (8) we get
\[
0 \leq \varphi''_1(x_1) = \varphi'_1(x_1) A(x_1) \quad \text{and} \quad 0 \geq \varphi''_1(x_2) = \varphi'_1(x_2) A(x_2).
\]
Since \( \varphi'_1(x) > 0 \) in \((-1, 1)\), it must be \( A(x_1) \geq 0 \) and \( A(x_2) \leq 0 \). By this we deduce that \( x_1 \geq \overline{x} \) but this produces the contradiction \( A(x_2) > 0 \).

The basic idea in the proof of Proposition 1 is to study the sign of the \( \varphi''_1 \) expression that can be deduced from (3). A direct inspection of this expression shows that if \( f'_X(x) > 0 \) for all \( x \in (0, 1) \), then \( \varphi_1 \) is concave in \((0, 1)\). This also tells us that the concavity study of \( \varphi_1 \) in the unimodal case presents all the main difficulties that one could find in the multimodal one.
Hypotheses (i) and (ii) of Proposition 1 requiring an a priori estimate of \( \xi_1 \) are, in general, difficult to handle. Here we state a sufficient condition for their validity.

**PROPOSITION 2.** Let \( X \in \mathcal{H}(D) \) and suppose there exists \( n_0 \geq 4 \) (even) such that \( f_X \) is differentiable \( n_0 \) times in \((-1, 1)\), and

\[
\frac{d^{3n_0}}{dx^{3n_0}} \ln(f_X(0)) = \cdots = \frac{d^{n_0-1}}{dx^{n_0-1}} \ln(f_X(0)) = 0; \quad \frac{d^n}{dx^n} \ln(f_X(0)) \neq 0.
\]

If \( \frac{d^3}{dx^3} \ln(f_X(x)) < 0 \) in \((0, 1)\), then \( \varphi_1 \) is concave in \([0, 1]\).

**PROOF.** We show that function \( A \) in (4) satisfies (i) and (ii) of Proposition 1. The assumption \( \frac{d^3}{dx^3} \ln(f_X(x)) < 0 \) implies that the function \( A \) is strictly increasing in \((0, 1)\). This readily implies (ii) of Proposition 1.

To check (i), we assume by contradiction that \( A(0) \geq 0 \). Thus, by the monotonicity of \( A \) and from (5) in the proof of Proposition 1, the first NLPC \( \varphi_1 \) associated to \( X \) satisfies

\[
\text{if } x_1 \in (0, 1) : \varphi''_1(x_1) = 0 \implies \varphi'''_1(x_1) > 0. \tag{9}
\]

Furthermore, we have that \( \varphi''_1(0) = 0 \) and \( \limsup_{x \to 1^-} \varphi''_1(x) < 0 \). If \( A(0) > 0 \), then \( \varphi'''_1(0) > 0 \) hence \( \varphi''_1(x) > 0 \) in a right neighborhood of \( x = 0 \). Hence, since \( \limsup_{x \to 1^-} \varphi''_1(x) < 0 \), (9) gives a contradiction.

Assume now that \( A(0) = 0 \), then \( \varphi'''_1(0) = 0 \). Differentiating in (6) we get \( \varphi''_1(0) = 0 \) for \( i = 2, \ldots, n_0 \) and

\[
\varphi^{n_0+1}_1(0) = \varphi'_1(0)A^{(n_0-2)}(0) = -\varphi'_1(0)\frac{d^n}{dx^n} \ln(f_X(x))(0) > 0,
\]

where the fact that \( A^{(n_0-2)}(0) = -\frac{d^n}{dx^n} \ln(f_X(x))(0) > 0 \) follows from the monotonicity of \( A \). We conclude that \( \varphi''_1(x) \) is positive in a left neighborhood of \( x = 0 \) and the contradiction comes arguing as for the case \( A(0) > 0 \).

We present now two families of distributions to which Proposition 2 applies.

**EXAMPLE 1.** For the one parameter family of centered, scaled and symmetric beta \((ess \beta(r)) \) on \( D = (-1, 1) \)

\[
f_X(x, r) = K_r \left(1 - x^2\right)^r \quad r \in (0, +\infty), \quad K_r = \left[\int_{-1}^{1} \left(1 - x^2\right)^r dx\right]^{-1} \tag{10}
\]

assumption \( (H1) \) has been tested in [6, Example 15] and \( (H2) \) holds. Some computations give for all \( x \in (0, 1) \)

\[
\frac{d^3}{dx^3} \ln(f_X(x, r)) = -\frac{4rx(2x^2 + 3)}{(1 - x^2)^3} < 0; \quad \frac{d^4}{dx^4} \ln(f_X(x, r)) = -\frac{12r(x^4 + 6x^2 + 1)}{(x^2 - 1)^4} \neq 0.
\]

Hence, Proposition 2 and, in turn, Proposition 1 applies.
Another family of distributions to which Proposition 2 applies is the Generalized Normal truncated distribution on $D = (-1, 1)$:

$$f_X(x) = K_m e^{-x^2 m}, \quad m \in \mathbb{N}, m \geq 2, K_m > 0.$$ 

Here, (H1) follows from [6, Theorem 5] and (H2) holds.

Next example shows that the assumptions of Proposition 2 are not necessary.

**EXAMPLE 2.** Consider the “Logistic truncated distribution”:

$$f_X(x) = \frac{(e + 1) e^x}{(e - 1)(1 + e^x)^2}, \quad x \in [-1, 1]. \quad (11)$$

Since $\frac{d^2}{dx^2} \ln(f_X(x)) > 0$, Proposition 2 does not apply. Anyway, as $f_X(1) \neq 0$ and $\varphi_1 \in W^{1,2}$, it holds:

$$\xi_1^{-1} = \int_{-1}^{1} \varphi_2^2(x) f_X(x) dx / \int_{-1}^{1} (\varphi_1')^2(x) f_X(x) dx \leq f_X(0) \max_{u \in W^{1,2}} \int_{-1}^{1} u^2(x) dx / \int_{-1}^{1} (u')^2(x) dx = f_X(0) 4 / f_X(1) \pi^2$$

hence $\xi_1 \geq e \pi^2 / (e + 1)^2$. In turn, this implies

$$A(0) = -\frac{d^2}{dx^2} \ln(f_X(0)) - \xi_1 \leq \frac{1}{2} - \frac{e \pi^2}{(e + 1)^2} < 0$$

and, jointly with the fact that $A'(x) < 0$ in $(0, 1)$, it allows to apply Proposition 1. Similarly one can treat the Standard Normal truncated distribution:

$$f_X(x) = K e^{-x^2/2}, \quad K > 0, \quad x \in [-1, 1]$$

having zero third logarithmic derivative. Note that for the above distributions assumption (H1) follows from [6, Theorem 5], while (H2) is easily verified.

Under the assumptions of Proposition 1 we are able to obtain a comparison principle for unimodal symmetric distributions, extending a result obtained in [10] for the uniform one. We note that this result does not seem easily extendible to the asymmetric case.

**PROPOSITION 3.** Let $X$ and $Y$ be in $\mathcal{H}(D)$. If $X$ satisfies the assumptions of Proposition 1, $f_X$ intersects $f_Y$ once in $(0, 1)$ and $f_X(0) > f_Y(0)$, then $\lambda_1^{f_X} < \lambda_1^{f_Y}$ where $\lambda_1^{f_X}$ and $\lambda_1^{f_Y}$ are the variances of the first NLPC of $X$ and $Y$, respectively.

**REMARK 1.** The hypothesis of Proposition 3 can be relaxed assuming that $f_X(x) \geq f_Y(x)$ for every $x \in [0, x_1]$, being $x_1$ the intersection point. Furthermore, a similar statement holds if $f_X$ intersects $f_Y$ $(2N + 1)$ times in $(0, 1)$ $(N \geq 0)$. More precisely, named $x_i$ $(i = 1, ..., 2N + 1)$ the intersection points, if $f_X(0) > f_Y(0)$ and $\int_{x_{2k}}^{x_{2k+2}} f_X = \int_{x_{2k}}^{x_{2k+2}} f_Y$, $\forall 0 \leq k \leq N$, where $x_0 = 0$ and $x_{2N+2} = 1$, then one still gets the comparison principle.

**PROOF.** By the last assumption, there must exist $x_1 \in (0, 1)$ such that $f_X(x) > f_Y(x)$ on $[0, x_1]$, and $f_X(x) < f_Y(x)$ on $(x_1, 1)$. Let $\varphi_1 \in W^{1,2}_{X}$ be the first NLPCs
transformation associated to $f_X$. Since $\varphi_1 \in C^1(-1, 1)$ is concave in $(0, 1)$ its first derivative $\varphi'_1$ is decreasing there. Thus there exists $\lim_{x \to 1^-} \varphi'_1(x)$ which, being $\varphi'_1$ positive, must be finite and, in particular, $\varphi_1 \in W^{1,2}$. By this, $\lim_{x \to 1^-} \varphi_1(x)$ is finite too. We have $\varphi_1 \in W^{1,2} \subset \overline{W_Y^{1,2}}$, where the embedding is due to the boundedness of $f_Y$. The strict monotonicity of $\varphi_1$ (see Lemma 1), by which $\varphi_1^2(x)$ is strictly increasing on $[0, 1]$, gives

$$
\int_{-1}^1 \varphi_1^2(x) (f_X(x) - f_Y(x))dx = 2 \int_{0}^1 \varphi_1^2(x) (f_X(x) - f_Y(x))dx
$$

$$
= 2 \int_{0}^{x_1} \varphi_1^2(x) (f_X(x) - f_Y(x))dx + 2 \int_{x_1}^{1} \varphi_1^2(x) (f_X(x) - f_Y(x))dx
$$

$$
< 2 \int_{0}^{x_1} \varphi_1^2(x_1) (f_X(x) - f_Y(x))dx + 2 \int_{x_1}^{1} \varphi_1^2(x_1) (f_X(x) - f_Y(x))dx
$$

$$
= \varphi_1^2(x_1) \int_{-1}^1 (f_X(x) - f_Y(x))dx = 0
$$

that is

$$
\int_{-1}^1 \varphi_1^2(x) f_X(x)dx < \int_{-1}^1 \varphi_1^2(x) f_Y(x)dx. \tag{12}
$$

Since by Proposition 1 transformation $\varphi_1$ is concave on $[0, 1]$, it follows that $(\varphi'_1(x))^2$ is decreasing on $[0, 1]$. Thus, in a completely analogous way, we deduce

$$
\int_{-1}^1 (\varphi'_1(x))^2 f_X(x)dx \geq \int_{-1}^1 (\varphi'_1(x))^2 f_Y(x)dx. \tag{13}
$$

By (12) and (13), we finally conclude that

$$
\lambda_{1X}^f = \frac{\int_{-1}^1 \varphi_1^2(x) f_X(x)dx}{\int_{-1}^1 (\varphi'_1(x))^2 f_X(x)dx} < \max_{\varphi \in W_Y^{1,2}} \frac{\int_{-1}^1 \varphi^2(x) f_Y(x)dx}{\int_{-1}^1 (\varphi'(x))^2 f_Y(x)dx} = \lambda_{1Y}^f.
$$

Since, under the assumptions of Proposition 3, it holds $\mathbb{E} [X^2] < \mathbb{E} [Y^2]$ we conclude that for the set of unimodal symmetric distributions considered, the variance ordering is preserved passing to the corresponding first NLPCs.

**EXAMPLE 3.** Consider the css$\beta(r)$ family (10). A direct inspection of $K_r$ gives $r_2 > r_1$ if and only if $K_{r_2} > K_{r_1}$, $r_1, r_2 \in \mathbb{R}_+$. Thus $f_X(0, r) = K_r$ is increasing with respect to $r$. Furthermore, when $r$ varies, the $f_X(x, r)$ intersect themselves once. On the other hand, by Example 1, we know that $f_X(x, r)$ satisfies the assumptions of Proposition 1 for all $r$. Hence Proposition 3 applies and, setting $\lambda_{1}^f := \lambda_{1}^{f_X(x, r)}$, we get $r_2 > r_1$ if and only if $\lambda_{1}^2 < \lambda_{1}^1, \forall r \in \mathbb{R}_+$.

### 3 An Application

In [6] a goodness-of-fit test for uniform distributions against unimodal distributions, based on a comparison result proved in [10], was given. Proposition 3 and Remark
1 allow to characterize all the distributions involved only by the knowledge of $\lambda_1$, permitting to generalize such a test procedure.

As explanatory example, we test $X \in \mathcal{N}([-1,1])$ is Wigner (that is $css/3 \beta(1/2)$, see (10)) against any other unimodal symmetric distribution and we state the hypothesis $H_0: \lambda_1 = \lambda_1^W$ against $H_1: \lambda_1 \neq \lambda_1^W$, where $\lambda_1^W = 0.28096$ is the variance of the first NLPC of a Wigner distribution on $[-1,1]$ computed by the package SLEIGN2 ([11]).

This last computation is theoretically supported by the following

**PROPOSITION 4.** A Wigner r.v. $X$ admits NLPCs $\varphi_j(X) = ce_j(\arccos(X), q_j)$, $j \in \mathbb{N}\setminus\{0\}$ where the $ce_j(\theta, q)$ are Mathieu functions (see [1] and [9]). Furthermore $\lambda_1 = (2a_1(q_1))^{-1}$, where $a_1(q)$ is a characteristic value and $q_1$ is the unique solution of $a_1(q) = 2q$

**PROOF.** We recall that (see [1] and [9]) the $2\pi$-periodic even solutions of the Mathieu equation:

$$\frac{d^2}{dx^2}u(x) + (a - 2q \cos(2\theta))u(x) = 0, \quad a, \theta, q \in \mathbb{R}$$

are called (even) Mathieu functions, usually indicated with $ce_j(\theta, q)$, $j \geq 1$. They can be expressed in uniformly convergent Fourier series of cosines where the coefficients can be determined only when $a$ belongs to the set of the so called characteristic value $a_j(q)$ of the Mathieu equation. For the Wigner distribution, problem (3) can be written as

$$\begin{aligned}
&\begin{cases}
(x^2 - 1)u''(x) + xu'(x) = \xi \left(1 - x^2\right)u(x) & x \in (-1,1), \quad \xi \in \mathbb{R}_+
\end{cases}
\end{aligned}
$$

By setting $x = \cos(\theta)$ and $z(\theta) = u(\cos(\theta))$, the equation in (15) becomes (14), but with $\theta \in (0, \pi)$ and $a = 2q = \xi/2$. Each solution of (15) can be extended to $\mathbb{R}$ in a $2\pi$-periodic even way, hence becoming one of the Mathieu functions $ce_j(\theta, q)$ (if $z(\theta)$ solves (14) the same holds for $z(\theta + k\pi)$, $k \in \mathbb{Z}$). We prove that, fixed $j$, for each family $ce_j(\theta, q)$, depending on $q \in \mathbb{R}_+$, there exists a unique value $q_j$ such that $ce_j(\arccos(x), q_j)$, with $x \in (-1,1)$, solves problem (15). By construction, the $ce_j(\arccos(x), q_j)$ satisfy the boundary conditions in (15), for every $j \geq 1$ and $q \in \mathbb{R}_+$. Furthermore, by the continuity of $a_j(q)$ and $a_j(0) = q^2$, $a_j(q) \sim -2q + O(q^{1/2})$ as $q \to +\infty$, we get the existence, for every $j \geq 1$, of at least a solution $q_j$ of $a_j(q) = 2q$.

To each $q_j$ it corresponds a solution $ce_j(\arccos(x), q_j)$ of (14) with $\xi = \xi_j = 2a_j(q_j)$. Recalling that each $ce_j(\theta, q)$ has exactly $j$ zeros in $(0, \pi)$, independently on $q$ (see [9], p. 234), the uniqueness of $q_j$, for every $j \geq 1$, follows by the simplicity of each $\xi_j$ combined with the fact that two eigenfunctions can not have the same number of zeroes in $(-1,1)$. Finally, the completeness in $W^1_{1,2}$ of the set $ce_j(\arccos(x), q_j)$ follows by standard theory of compact operators on Hilbert spaces.

To define the critical region of this test, we introduce the statistic $\delta_n = \sqrt{n} |\hat{\lambda}_1 - \lambda_1^W|$, where $\hat{\lambda}_1$ is a suitable estimate of $\lambda_1$ from a sample of size $n$ (see [6]). We obtain the critical values by a Monte Carlo calculation based on five hundred replications.

Some numerical experiments to study the level and the power of the test proposed are carried out, having chosen as alternatives the $css/3 \beta(r)$ family (10) and the Truncated Normal distribution $\mathcal{N}^T(0, \sigma)$ on $[-1,1]$. Sample sizes $n = 100$, 200 and 500 were
considered. Testing at the level $\alpha = 0.1$, results obtained from five hundred simulations, are compared with the ones by the Kolmogorov-Smirnov and the Chi-square test. The substantially good performances of the test based on $\delta_n$ can be deduced from Table 1.

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Table 1: Estimated level and power in comparison with the Kolmogorov-Smirnov (K-S) and the Chi-square ($\chi^2$) test ($\alpha = 0.1$).

References


