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An Improved Existence And Uniqueness Criterion For Periodic Solutions Of A Duffing Type p-Laplacian Equation*

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Abstract

In this study, we investigate a Duffing type p-Laplacian equation. Some new criteria for guaranteeing the existence and uniqueness of periodic solutions of this equation are given by using the Manásevich–Mawhin continuation theorem. Our results extend and improve some known results from the literature.

1 Introduction

In this note, we consider the existence and uniqueness of periodic solution of a Duffing type p-Laplacian equation as follows:

$$(\varphi_p(x'(t)))' + Cx'(t) + g(t, x(t)) = e(t), \tag{1}$$

where $p > 1$, $\varphi_p : \mathbb{R} \to \mathbb{R}$, $\varphi_p(s) = |s|^{p-2}s$ is a one-dimensional p-Laplacian; C is a constant; $g, e \in C(\mathbb{R}, \mathbb{R})$, $\int_0^T e(t)dt = 0$ and $T > 0$.

As is known, the Duffing equation can be derived from many fields, such as physics, mechanics and engineering technique fields, and an important question is whether this equation can support periodic solutions. During the past few years, many researchers discussed the periodic solutions of Duffing equation, see e.g., [3, 4, 6]. Recently, Wang and Ge [2] discussed the existence of periodic solutions of Duffing equation like (1) by using polar coordinates. Very recently, Zhang and Li [1] studied the periodic solutions of (1) by using Mánasevich–Mawhin continuation theorem, and got some results as follows:

THEOREM 1. ([1]) Suppose that there exist $K > 0$ and $M > 0$ such that:
(A_1) $[g(t, u_1) - g(t, u_2)](u_1 - u_2) < 0$, for all u_1, u_2 and $t \in \mathbb{R}$ with $u_1 \neq u_2$,
(A_2) $xg(t, x) < 0$, for all $|x| > 0$ and $t \in \mathbb{R}$,

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(A$_3$) $2^{2-p}MT^p < 1$ and $g(t,x) \geqslant -M|x|^{p-1} - K$, for all $x \geqslant 0$ and $t \in \mathbb{R}$.
Then (1) has a unique T-periodic solution.

THEOREM 2. ([1]) Suppose that there exist $K > 0$ and $M > 0$ such that:
(A$_1$) $[g(t,u_1) - g(t,u_2)](u_1 - u_2) < 0$, for all u_1, u_2 and $t \in \mathbb{R}$ with $u_1 \neq u_2$,
(A$_2$) $xg(t,x) < 0$, for all $|x| > 0$ and $t \in \mathbb{R}$,
(A$_3'$) $2^{2-p}MT^p < 1$ and $g(t,x) \leqslant M|x|^{p-1} + K$, for all $x \leqslant 0$ and $t \in \mathbb{R}$.
Then (1) has a unique T-periodic solution.

However, upon examining the proofs of the above-mentioned Theorems 1 and 2, we found that the conditions (A_3) and (A_3') can be abandoned, and the condition (A_2) also can be replaced by a weaker one.

In this paper, we reconsider the periodic solutions of (1). The main purpose of this paper is to establish some new criteria for guaranteeing the existence and uniqueness of periodic solution of (1). Our results extend and improve the above-mentioned Theorems 1 and 2 in [1].

The following notation will be used throughout the rest of this paper.

$$|x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x'|_\infty = \max_{t\in[0,T]} |x'(t)|, \quad |x|_k = \left(\int_0^T |x(t)|^k dt\right)^{1/k}.$$

Set

$$C_T^1 := \{x \in C^1(\mathbb{R},\mathbb{R}) : x(t+T) = x(t)\}$$

and

$$C_T := \{x \in C(\mathbb{R},\mathbb{R}) : x(t+T) = x(t)\},$$

which are two Banach spaces with the norms

$$||x||_{C_T^1} = \max\{|x|_\infty, |x'|_\infty\}, \quad ||x||_{C_T} = |x|_\infty.$$

Now our main theorem is in the following:

THEOREM A. Assume that there exists $d \geqslant 0$ such that:
(H$_1$) $[g(t,u_1) - g(t,u_2)](u_1 - u_2) < 0$, for all u_1, u_2 and $t \in \mathbb{R}$ with $u_1 \neq u_2$,
(H$_2$) $xg(t,x) < 0$, for all $|x| > d$ and $t \in \mathbb{R}$.
Then (1) has a unique T-periodic solution.

2 Lemmas

For the periodic boundary value problem

$$(\varphi_p(x'(t)))' = h(t,x,x'), x(0) = x(T), x'(0) = x'(T), \tag{2}$$

where $h \in C(\mathbb{R}^3, \mathbb{R})$ is T-periodic in the first variable. We have the following results.

LEMMA 1. (Manásevich–Mawhin [5]). Let $B = \{x \in C_T^1 : \|x\| < r\}$ for some $r > 0$. Suppose the following two conditions hold:

(i) For each $\lambda \in (0,1)$ the problem $(\varphi_p(x'(t)))' = \lambda h(t,x,x')$ has no solution on ∂B.

(ii) The continuous function F defined on $\mathbb{R}$ by $F(a) = \frac{1}{T}\int_0^T h(t,a,0)dt$ is such that $F(-r)F(r) < 0$.

Then the periodic boundary value problem (2) has at least one T-periodic solution on $\bar{B}$.

The following Lemma will help us for obtaining the uniqueness of periodic solution of (1).

LEMMA 2. ([1]) Suppose *(H_1)* holds. Then (1) has at most one T-periodic solution.

3 Proof of Theorem A

Now we are in the position to present the proof of Theorem A.

By Lemma 2, we know that (1) has at most one T-periodic solution. Thus it suffices to show that (1) has at least one T-periodic solution. To do this, Lemma 1 will be applied.

Consider the homotopic equation of (1):

$$(\varphi_p(x'(t)))' + \lambda C x'(t) + \lambda g(t,x(t)) = \lambda e(t), \lambda \in (0,1). \tag{3}$$

First, we prove that the set of T-periodic solutions of (3) are bounded in C_T^1. Let $S \subset C_T^1$ be the set of T-periodic solutions of (3). If $S = \emptyset$, the proof is ended. Suppose $S \neq \emptyset$, and let $x \in S$. Noticing that $x(0) = x(T)$, $x'(0) = x'(T)$, $\varphi_p(0) = 0$, and $\int_0^T e(t)dt = 0$, it follows from (3) that

$$\int_0^T g(t,x(t))dt = 0,$$

which implies that there exists $t_0 \in [0,T]$ such that

$$g(t_0, x(t_0)) = 0. \tag{4}$$

By (H_2), (4) implies

$$|x(t_0)| < d. \tag{5}$$

Then for any $t \in [t_0, t_0+T]$, we have

$$|x(t)| = \left| x(t_0) + \int_{t_0}^t x'(s)ds \right| < d + \int_{t_0}^{t_0+T} |x'(s)|ds = d + \int_0^T |x'(s)|ds,$$

which leads to

$$|x|_\infty = \max_{t\in[t_0,t_0+T]} |x(t)| < d + |x'|_1. \tag{6}$$

Now define $E_1 = \{t : t \in [0, T], |x(t)| > d\}, E_2 = \{t : t \in [0, T], |x(t)| \leqslant d\}$. Then multiplying $x(t)$ and (3) and integrating from 0 to T, we have by (H_2) that

$$\begin{aligned}
\int_0^T |x'(t)|^p dt &= -\int_0^T (\varphi_p(x'(t)))' x(t) dt \\
&= \lambda \int_0^T g(t, x(t)) x(t) dt - \lambda \int_0^T e(t) x(t) dt \\
&= \lambda \int_{E_1} g(t, x(t)) x(t) dt + \lambda \int_{E_2} g(t, x(t)) x(t) dt - \lambda \int_0^T e(t) x(t) dt \\
&\leq \lambda \int_{E_2} g(t, x(t)) x(t) dt - \lambda \int_0^T e(t) x(t) dt \\
&\leq \int_{E_2} |g(t, x(t))||x(t)| dt + \int_0^T |e(t)||x(t)| dt \\
&\leq \left(\max_{t \in [0,T], |x| \leq d} |g(t, x)| + |e|_\infty \right) T |x|_\infty.
\end{aligned}$$

Let $M_0 = \left(\max_{t \in [0,T], |x| \leq d} |g(t, x)| + |e|_\infty \right) T$. Then we obtain

$$|x'|_p \leq M_0^{1/p} |x|_\infty^{1/p}. \tag{7}$$

Let $q > 1$ such that $1/p + 1/q = 1$. Then by Hölder inequality we have

$$|x'|_1 \leq |x'|_p |1|_q = T^{1/q} |x'|_p. \tag{8}$$

By (6), (7) and (8), we can get

$$|x'|_1 \leq T^{1/q} M_0^{1/p} (d + |x'|_1)^{1/p},$$

which yields that there exists $M_1 > 0$ such that $|x'|_1 < M_1$ since $p > 1$, and this together with (6) implies that $|x|_\infty < d + M_1$.

Meanwhile, there exists $\hat{t}_0 \in [0, T]$ such that $x'(\hat{t}_0) = 0$ since $x(0) = x(T)$. Then by (3) we have, for $t \in [\hat{t}_0, \hat{t}_0 + T]$,

$$\begin{aligned}
|\varphi_p(x'(t))| &= \left| \int_{\hat{t}_0}^t (\varphi_p(x'(s)))' ds \right| \\
&= \lambda \left| \int_{\hat{t}_0}^t (Cx'(s) + g(s, x(s)) + e(s)) ds \right| \\
&\leq \int_0^T (C|x'(s)| + |g(s, x(s))| + |e(s)|) ds \\
&< CM_1 + (G + |e|_\infty) T,
\end{aligned}$$

where $G = \max\{|g(t, x)| : t \in [0, T], |x| \leqslant d + M_1\}$. So we obtain

$$|x'|_\infty = \max_{t \in [0,T]} \{|\varphi_p(x'(t))|^{1/(p-1)}\} < (CM_1 + (G + |e|_\infty) T)^{1/(p-1)}.$$

Let $M = \max\{d + M_1, (CM_1 + (G + |e|_\infty)T)^{1/(p-1)}\}$. Then $\|x\| < M$.

Second, we prove the existence of T-periodic solutions of (1). Set

$$h(t, x(t), x'(t)) = -Cx'(t) - g(t, x(t)) + e(t), \tag{9}$$

then (3) is equivalent to the following equation

$$(\varphi_p(x'(t)))' = \lambda h(t, x(t), x'(t)), \lambda \in (0, 1). \tag{10}$$

Set

$$B = \{x : x \in C_T^1, \|x\| < r\} \quad \text{where } r \geq M, \tag{11}$$

by (9), we know that (10) has no solution on ∂B as $\lambda \in (0, 1)$, so condition (i) of Lemma 1 is satisfied. By the definition of F in Lemma 1 we get

$$F(a) = \frac{1}{T}\int_0^T h(t, a, 0)dt = \frac{1}{T}\int_0^T (e(t) - g(t, a))dt = -\frac{1}{T}\int_0^T g(t, a)dt.$$

This together (H_2) yields that $F(r)F(-r) < 0$, i.e., condition (ii) of Lemma 1 is satisfied. Therefore, it follows from Lemma 1 that there exists a T-periodic solution $x(t)$ of (1). This completes the proof.

4 Remarks

It is easy to see that Theorem A in this study holds under weaker conditions than Theorems 1 and 2 in [1].

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