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Entire Functions That Share Values With Their Derivatives*

Feng Lü†

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Abstract

In this paper, we study a uniqueness problem of entire functions that share finite values with their derivatives. We deduce a theorem which generalizes some previous results given by Rebel and Yang [13], Jank, Mues and Volkmann [7] and Chang and Fang [3].

1 Introduction

Let f and g be two non-constant entire functions, and let $a,\ b$ be two finite complex numbers. If $g(z)-b=0$ whenever $f(z)-a=0$, then we denote this condition by $f(z)=a\Rightarrow g(z)=b$. If $f(z)=a\Rightarrow g(z)=a$ and $g(z)=a\Rightarrow f(z)=a$, we denote it by $f(z)=a\Leftrightarrow g(z)=a$ and say that f and g share a IM (ignoring multiplicity). Provided that $f-a$ and $g-a$ have the same zeros with the same multiplicities, we denote it by $f(z)=a\rightleftharpoons g(z)=a$ and say that f and g share a CM (counting multiplicity). If $f(z)=a\Rightarrow g(z)=b$ and the multiplicity of the zero z of $g-b$ is greater than or equal to that of the zero z of $f-a$, then we denote this condition by $f(z)-a=0\rightarrow g(z)-b=0$. In what follows, we assume that the reader is familiar with the basic notation and results in the Nevanlinna value distribution theory, as found in [6, 16].

The subject on sharing values between two meromorphic functions was studied for almost 80 years. Meanwhile, a number of outstanding results have been obtained (see [8, 9, 18]).

In 1977, Rubel and Yang [13] first studied the problem of sharing values between entire functions and their derivatives. They proved that if a non-constant entire function f and its first derivative f' share two distinct finite numbers $a,\ b$ CM, then $f=f'$. Since then, shared value problems, have been studied by many authors and a number of profound results have been obtained (see [1, 11, 15]).

In 1986, Junk et al.[7] studied similar problems and proved the following results.

THEOREM A. Let f be a non-constant entire function, and let a be a non-zero finite constant. If f, f' and f'' share the value a CM, then $f=f'$.

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†Department of Mathematics, China University of Petroleum, Dongying, Shandong, 257061, P. R. China.

THEOREM B. Let f be a non-constant entire function, and let a be a non-zero finite constant. If $f = a \Leftrightarrow f' = a$, $f = a \rightarrow f'' = a$, then $f = f'$.

In 2002, Chang and Fang [3] improved Theorem B and obtained the following result.

THEOREM C. Let f be a non-constant entire function, let a, c be two non-zero constants. If $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f''(z) = c$, then $f(z) = Ae^{\frac{cz}{a}} + \frac{ac-a^2}{c}$ or $f(z) = Ae^{\frac{cz}{a}} + a$, where A is a non-zero constant.

It is natural to ask what will happen if f'' is replaced by the k-th derivative $f^{(k)}$? It follows from the hypothesis of Theorem C that $f' - a$ has simple zeros only. However, if f'' is replaced by the k-th derivative $f^{(k)}$ $(k \geq 3)$ in Theorem C, we cannot deduce the property that $f' - a$ only has simple zeros. Thus it does not seem that we can solve this problem by similar methods. In this work, we use the theory of normal family to deal with the problem and derive the following result, which is an improvement of Theorem C.

THEOREM 1. Let f be a non-constant entire function, let a, c be two non-zero constants. If $f(z) = a \Rightarrow f'(z) = a$, $f'(z) = a \Rightarrow f'''(z) = c$, then $f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}$ or $f(z) = Ae^{\lambda z} + a$, where A is a non-zero constant and $\lambda^2 = \frac{c}{a}$.

REMARK 1. Recently, Chang and Fang [4] used a different way to solve the problem. They replaced the assumption $f'(z) = a \Rightarrow f'''(z) = c$ by $f'(z) = a \Rightarrow f^{(k)} = a$ and deduced the similar conclusion. But, their method are complicated. In contrast, our method is simple and easier to understand. This is the point of this work.

REMARK 2. If the hypothesis "$f'(z) = a \Rightarrow f'''(z) = c$" is replaced by "$f(z) = a \Rightarrow f'''(z) = c$", the conclusion is not generally true, we give the following negative example.

EXAMPLE. Let $f(z) = 1 + 6e^{3z} + 2e^{3z/2}$. Then $f'(z) = 18e^{3z} + 3e^{3z/2}$ and $f'''(z) = 162e^{3z} + (27/4)e^{3z/2}$. One can easily check that $f(z) = 1 \Rightarrow f'(z) = 1$ and $f(z) = 1 \Rightarrow f'''(z) = 63/4$. But f does not satisfy the conclusion of Theorem 1.

2 Some Lemmas

We state several preparatory Lemmas.

LEMMA 1. [2] Let f be an entire function, and let M be a positive number. If $f^\sharp(z) \leq M$ for any $z \in C$, then f is of exponential type.

LEMMA 2. [14] Let $\mathcal{F}$ be a family of meromorphic functions in domain D, then $\mathcal{F}$ is normal in D if and only if the spherical derivatives of functions $f \in \mathcal{F}$ are uniformly bounded on compact subsets of D.

LEMMA 3. [17] Let f be a non-constant entire function of finite order, and let a be a non-zero constant. If f and f' share a CM, then

$$\frac{f' - a}{f - a} = c,$$

for some non-zero constant c.

LEMMA 4. Let f be a transcendental entire function with $\rho(f) \leq 1$, and let $a,\ c$ be two non-zero constants. If $f(z) = 0 \Rightarrow f'(z) = a,\ f'(z) = a \Rightarrow f'''(z) = c$ and $N\left(r, \frac{f}{f'-a}\right) = S(r, f)$, then $f(z) = Ae^{\lambda z} - \frac{a}{\lambda}$, where A is a constant and $\lambda^2 = \frac{c}{a}$.

PROOF. Suppose that 0 is a Picard value of f. Noting that $\rho(f) \leq 1$, we can set $f(z) = Ae^{\lambda z}$, where A, λ are two non-zero constants. Thus

$$S(r,f) = N\left(r, \frac{f}{f'-a}\right) = N\left(r, \frac{1}{f'-a}\right) = N\left(r, \frac{1}{Ae^{\lambda z}-a}\right) = T(r,f) + S(r,f),$$

a contradiction.

In the following, we assume that 0 is not a Picard value of f. It follows from $f(z) = 0 \Rightarrow f'(z) = a$ that f only has simple zeros. Let

$$\varphi = \frac{af''' - cf'}{f}. \tag{1}$$

It is obvious that φ is an entire function. By the lemma of logarithmic derivative, we have

$$T(r,\varphi) = m(r,\varphi) = S(r,f).$$

We now distinguish the following two cases.

Case 1. $\varphi = 0$.

Then, $af''' = cf'$. By solving the differential equation, we obtain

$$f(z) = Ae^{\lambda z} + Be^{-\lambda z} + C_0, \tag{2}$$

where A, B, C_0 are constants.

If $AB \neq 0$, combining $f(z) = 0 \Rightarrow f'(z) = a$ and (2) yields that $f(z) = 0 \rightleftharpoons f'(z) = a$. Then, it follows from Lemma 2 that $\frac{f'-a}{f} = c_1$, where c_1 is a constant. Furthermore, with the above differential equation, we deduce that $f(z) = Ae^{\lambda z} - \frac{a}{\lambda}$, which contradicts (2).

If $AB = 0$. Without loss of generality, we assume $A \neq 0$. Then, it is easy to deduce that $f(z) = Ae^{\lambda z} - \frac{a}{\lambda}$ and $\lambda^2 = \frac{c}{a}$.

Case 2. $\varphi \neq 0$.

Rewriting (1) as $f = \frac{af'''-cf'}{\varphi}$. A routine calculation leads to

$$\left[1 + c\left(\frac{1}{\varphi}\right)'\right] f' = a\left(\frac{1}{\varphi}\right)' f''' - c\frac{1}{\varphi}f'' + a\frac{1}{\varphi}f^{(4)}. \tag{3}$$

Since φ is an entire function, then $1+c(\frac{1}{\varphi})' \neq 0$. From (3), we derive that $m\left(r, \frac{f'}{f'-a}\right) = S(r,f)$. Furthermore, we have

$$m\left(r, \frac{1}{f'-a}\right) \leq m\left(r, \frac{a}{f'-a}\right) + O(1) \leq m\left(r, \frac{f'}{f'-a} - 1\right) + O(1) = S(r,f) \tag{4}$$

and

$$T(r,f)=m(r,f)=m\left(r,\frac{af'''-cf'}{\varphi}\right)\le m(r,f')+S(r,f)$$
$$\le T(r,f')+S(r,f)\le T(r,f)+S(r,f),$$

which implies that

$$T(r,f)=T(r,f')+S(r,f).$$

The fact $N\left(r,\frac{f}{f'-a}\right)=S(r,f)$ leads to

$$N\left(r,\frac{1}{f}\right)=N\left(r,\frac{1}{f'-a}\right)+S(r,f).$$

Thus,

$$m\left(r,\frac{1}{f}\right)=T(r,f)-N\left(r,\frac{1}{f}\right)+O(1)=T\left(r,\frac{1}{f'-a}\right)-N\left(r,\frac{1}{f}\right)+O(1)$$
$$=m\left(r,\frac{1}{f'-a}\right)+N\left(r,\frac{1}{f'-a}\right)-N\left(r,\frac{1}{f}\right)+O(1)=S(r,f).$$

Let

$$\mu=\frac{f'-a}{f}. \tag{5}$$

Obviously, μ is an entire function. Noting that f is transcendental, then we have $\mu\ne 0$. With (5), it is easy to derive that

$$T(r,\mu)=m(r,\mu)\le m\left(r,\frac{a}{f}\right)+S(r,f)\le S(r,f).$$

Rewriting (5) as $f'=f\mu+a$ and differentiating it twice yields

$$f'''=f(\mu^3+3\mu\mu'+\mu'')+a(\mu^2+2\mu').$$

Combining $f(z)=0\Rightarrow f'''(z)=c$ and $m\left(r,\frac{1}{f}\right)=S(r,f)$ leads to $a(\mu^2+2\mu')=c$. Clearly, $c-2a\mu'\ne 0$. If μ is not a constant, then $2T(r,\mu)=T(r,c-2a\mu')\le T(r,\mu)+S(r,\mu)$, which indicates $T(r,\mu)=S(r,\mu)$, a contradiction. Hence μ must be a constant. By (5) and our assumption, we can easily deduce that $f(z)=Ae^{\lambda z}-\frac{a}{\lambda}$ and $\lambda^2=\frac{c}{a}$. Putting the form of f into (1) yields $\varphi=0$, a contradiction. This completes the proof of this lemma.

For our proof, we also need the following result. It can be easily obtained from [10, Theorem 1].

LEMMA 5. Let $\mathcal{F}$ be a family of holomorphic functions in a domain D, and let a, c be two non-zero constants. If for every $f\in\mathcal{F}$, $f(z)=0\Rightarrow f'(z)=a$ and $f'(z)=a\Rightarrow f'''=c$, then $\mathcal{F}$ is normal in D.

LEMMA 6. [18] Let f_1 and f_2 be two non-constant meromorphic functions satisfying

$$\overline{N}(r,f_i)+\overline{N}\left(r,\frac{1}{f_i}\right)=S(r),\quad i=1,2.$$

Then either

$$N_0(r,1;f_1,f_2)=S(r)$$

or there exist two integers s, t ($|s|+|t|>0$) such that

$$f_1^s f_2^t=1,$$

where $N_0(r,1;f_1,f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-point and $T(r)=T(r,f_1)+T(r,f_2)$, $S(r)=o(T(r))$ $(r\to\infty,\quad r\notin E)$ only depending on f_1 and f_2.

LEMMA 7. [12], [5, Theorem 4.1] Let f be an entire function of order at most 1 and k be a positive integer, then

$$m\left(r,\frac{f^{(k)}}{f}\right)=o(\log\ r),\quad \text{as}\quad r\to\infty.$$

3 Proof of Theorem 1

From the assumption, we derive that f is a transcendental entire function. Now, let us show that f is of exponential type. Set $g=f-a$. Then

$$g(z)=0\Rightarrow g'(z)=a,\quad g'(z)=a\Rightarrow g'''(z)=c.$$

Set $\mathcal{F}=\{g(z+w):w\in C\}$. Then $\mathcal{F}$ is a family of holomorphic functions on the unit disc $\triangle$. For any function $F(z)=g(z+w)$, we have

$$F(z)=0\Rightarrow F'(z)=a,\quad F'(z)=a\Rightarrow F'''(z)=c.$$

It follows from Lemma 5 that $\mathcal{F}$ is normal in $\triangle$. Then, by Lemma 2, there exists a positive number M satisfying $g^\sharp(z)\le M$ for all $z\in C$. Furthermore, with Lemma 1, we deduce that g is of exponential type. So, $\rho(f)=\rho(g)\le 1$. We also have

$$f(z)=a\Rightarrow f'(z)=a,\quad f'(z)=a\Rightarrow f'''(z)=c. \tag{6}$$

Suppose that a is a Picard value of f, then $f(z)=Ae^{\lambda z}+a$, where $A,\ \lambda$ are two non-zero constants. From (6) and the form of f, it is easy to obtain that $\lambda^2=\frac{c}{a}$.

In the following, we assume that a is not a Picard value of f.

Let

$$\varphi=\frac{af'''-cf'}{f-a}. \tag{7}$$

By (6), we derive that φ is an entire function. From Lemma 7, we have

$$\begin{aligned} T(r,\varphi) &= m(r,\varphi)=m\left(r,\frac{af'''-cf'}{f-a}\right)\\ &\le m\left(r,\frac{af'''}{f-a}\right)+m\left(r,\frac{cf'}{f-a}\right)+\log 2=o(\log r), \end{aligned}$$

which implies that φ reduces to a constant. Suppose $\varphi=c_2$.

We now consider into two cases.

Case 1. $c_2 \neq 0$.

We first analyze the property of the equation

$$f''' - \frac{c}{a}f' = \frac{c_2}{a}(f - a). \tag{8}$$

Noting that $g = f - a$, then

$$g''' - \frac{c}{a}g' = \frac{c_2}{a}g. \tag{9}$$

By (6) and (9), it is easy to deduce

$$g = 0 \Leftrightarrow g' = a \Rightarrow g''' = c. \tag{10}$$

We divide into two subcases as follows.

Subcase 1.1. $N_{(2}(r, \frac{1}{g'-a}) = S(r, g)$. Then

$$N(r, \frac{g}{g' - a}) \leq N_{(2}(r, \frac{1}{g' - a}) = S(r, g). \tag{11}$$

By Lemma 4, we get $f(z) = g(z) + a = Ae^{\lambda z} + a - \frac{a}{\lambda}$, where A is a non-zero constant and $\lambda^2 = \frac{c}{a}$. Thus $\varphi = c_2 = 0$, which is a contradiction.

Subcase 1.2. $N_{(2}(r, \frac{1}{g'-a}) \neq S(r, g)$.

It implies $g' - a$ has infinitely many multiple zeros. Again, we divide into two subcases.

Subcase 1.2.1. The equation $\lambda^3 - \frac{c}{a}\lambda - \frac{c_2}{a} = 0$ has a multiple zero.

Then, we deduce the multiplicity of the zero is two. Thus,

$$g(z) = (C_{11} + C_{12}z)e^{\lambda_1 z} + C_2 e^{\lambda_2 z}. \tag{12}$$

Suppose $C_{12} \neq 0$, then

$$g''(z) = (2\lambda_1 C_{12} + C_{11}\lambda_1^2 + C_{12}\lambda_1^2 z)e^{\lambda_1 z} + C_2\lambda_2^2 e^{\lambda_2 z}. \tag{13}$$

Let z_n be the multiple zero of $g' - a$. Then $g(z_n) = 0$ and $g''(z_n) = 0$. Combining (12) and (13) yields

$$2\lambda_1 C_{12} + C_{11}(\lambda_1^2 - \lambda_2^2) + C_{12}(\lambda_1^2 - \lambda_2^2)z_n = 0.$$

In view of $z_n \to \infty$, we derive $\lambda_1^2 = \lambda_2^2$ and $C_{12} = 0$, this is a contradiction.

Now, we assume $C_{12} = 0$. Then

$$g(z) = C_{11}e^{\lambda_1 z} + C_2 e^{\lambda_2 z}.$$

If $C_{11}C_2 \neq 0$, similarly as above, we conclude that $\lambda_1 = -\lambda_2$. So,

$$g(z) = C_{11}e^{\lambda_1 z} + C_2 e^{-\lambda_1 z}. \tag{14}$$

From (10) and (14), we derive that

$$g(z) = 0 \rightleftharpoons g'(z) = a, \tag{15}$$

which implies that $N_{(2}(r, \frac{1}{g'-a}) = 0$, a contradiction.

If $C_{11}C_2 = 0$, obviously, this is absurd.

Subcase 1.2.2. The equation $\lambda^3 - \frac{c}{a}\lambda - \frac{c_2}{a} = 0$ only has simple zeros.

From (9), we have

$$g(z) = c_1 e^{\lambda_1 z} + c_2 e^{\lambda_2 z} + c_3 e^{\lambda_3 z}. \tag{16}$$

If $c_1c_2c_3 = 0$, similarly as above, we can get a contradiction.

Suppose $c_1c_2c_3 \neq 0$. Let z_n be the multiple zero of $g' - a$. Then $g(z_n) = 0$, $g'(z_n) = a$ and $g''(z_n) = 0$. With (16), it is not difficult to deduce that

$$e^{\lambda_j z_n} = D_j \quad (1 \leq j \leq 3), \tag{17}$$

where $D_j \neq 0$ $(1 \leq j \leq 3)$ are constants. Assume that

$$f_j(z) = e^{\lambda_j z}/D_j \quad (1 \leq j \leq 3). \tag{18}$$

Noting that (18) and the fact that the multiplicity of a-points of g' is 2, we derive that

$$N_{(2}\left(r, \frac{1}{g'-a}\right) \leq 2N_0(r, 1; f_1, f_2).$$

So

$$N_0(r, 1; f_1, f_2) \neq S(r)$$

and

$$\overline{N}(r, f_i) + \overline{N}(r\frac{1}{f_i}) = S(r), \quad i = 1, 2.$$

Thus by Lemma 6, there exist two integers s_1, t_1 $(|s_1| + |t_1| > 0)$ such that

$$f_1^{s_1} f_2^{t_1} = 1.$$

Then $\lambda_1 s_1 + \lambda_2 t_1 = 0$, $\lambda_2 = -\frac{s_1}{t_1}\lambda_1$. Similarly, we can deduce that $\lambda_3 = -\frac{s_2}{t_2}\lambda_1$. Let $\lambda_1 = t_1 t_2 \lambda = p_1 \lambda$. Then

$$\lambda_2 = -s_1 t_2 \lambda = p_2 \lambda, \quad \lambda_3 = -s_2 t_1 \lambda = p_3 \lambda.$$

From the equation

$$\lambda^3 - \frac{c}{a}\lambda - \frac{c_2}{a} = 0, \tag{19}$$

we derive

$$p_1\lambda + p_2\lambda + p_3\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 0,$$

which implies that

$$p_1 + p_2 + p_3 = 0, \tag{20}$$

where p_1, p_2, p_3 are three integers.

By (20), we know there exist a positive integer and a negative integer in $\{p_1, p_2, p_3\}$. Without loss of generality, we assume $p_1 > 0$ and $p_2 < 0$. Noting that $p_2 \neq p_3$, we suppose $p_3 > p_2$.

Rewriting (16) as

$$g(z) = c_1 e^{p_1 \lambda z} + c_2 e^{p_2 \lambda z} + c_3 e^{p_3 \lambda z}. \tag{21}$$

Set

$$P(z) = c_1 z^{p_1} + c_2 z^{p_2} + c_3 z^{p_3} \tag{22}$$

and

$$Q(z) = \lambda[c_1 p_1 z^{p_1} + c_2 p_2 z^{p_2} + c_3 p_3 z^{p_3}]. \tag{23}$$

Then

$$g(z) = P(e^{\lambda z}) \quad \text{and} \quad g'(z) = Q(e^{\lambda z}). \tag{24}$$

From (10), we obtain that $g(z)$ only has simple zeros.

If $p_1 > p_3$, from (22) we have P has m simple roots and $m = p_1 - p_2$. It follows from (23) that $Q - a$ at most has m roots. Then, combining (10) and (24) leads to

$$g = 0 \rightleftharpoons g' = a,$$

which indicates that $N_{(2}\left(r, \frac{1}{g'-a}\right) = 0$, a contradiction.

If $p_1 < p_3$, similarly as above, we also deduce a contradiction.

Case 2. $c_2 = 0$.

Then

$$af''' = cf'.$$

Solving the above differential equation yields

$$f(z) = C_1 e^{\lambda_1 z} + C_2 e^{-\lambda_1 z} + C_0, \tag{25}$$

where C_0 is a constant. We divide into two cases.

Case 2.1. $C_1 C_2 \neq 0$.

With (25) and $f(z) = a \Rightarrow f'(z) = a$, it is not difficult to derive that

$$f(z) = a \rightleftharpoons f'(z) = a.$$

Thus, we deduce that $f(z) = Ae^{\lambda z} + a - \frac{a}{\lambda}$, which contradicts (25).

Case 2.2. $C_1 C_2 = 0$.

Without loss of generality, we suppose $C_2 = 0$. Then, after a simple calculation, we obtain the conclusion of Theorem 1.

Thus we complete the proof of the theorem.

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