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Weighted Morrey-Herz Spaces And Applications*

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Abstract

In this paper, we introduce the weighted Morrey-Herz spaces. We also obtain the necessary and sufficient conditions for the weighted Hardy-Littlewood mean operators to be bounded on these weighted Morrey-Herz spaces. Results proved in this paper can be viewed as significant refinement of several previously known results.

1 Introduction

Let $k \in Z$, $B_k = \{x \in \mathbf{R}^n : |x| \leq 2^k\}$, $D_k = B_k - B_{k-1}$ and let $\varphi_k = \varphi_{D_k}$ denote the characteristic function of the set D_k. Moreover, for a measurable function f on $\mathbf{R}^n$ and a non-negative weighted function $\omega(x)$, we write

$$\|f\|_{p,\omega} = \left(\int_{R^n} |f(x)|^p \omega(x) dx \right)^{1/p}.$$

In what follows, if $\omega \equiv 1$, then we will denote $L^p(R^n, \omega)$ (in brief $L^p(\omega)$) by $L^p(R^n)$. Let $\alpha \in \mathbf{R}^1$, $0 < p, q < \infty$ and $\lambda \geq 0$. The Morrey spaces $M_q^\lambda(R^n)$ is defined by [1] as follows:

$$M_q^\lambda(R^n) = \left\{ f \in L_{loc}^q(R^n) : \sup_{r>0, x \in R^n} \frac{1}{r^\lambda} \int_{|x-y|<r} |f(y)|^q dy < \infty \right\}, \tag{1}$$

and the homogeneous Herz space $\dot{K}_q^{\alpha,p}(R^n)$ is defined by [2] as follows:

$$\dot{K}_q^{\alpha,p}(R^n) = \{ f \in L_{loc}^q(R^n - \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(R^n)} < \infty \}, \tag{2}$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}(R^n)} = \left\{ \sum_{k \in Z} 2^{k\alpha p} \|f\varphi_k\|_q^p \right\}^{1/p}. \tag{3}$$

We can similarly define the non-homogeneous Herz space $K_q^{\alpha,p}(R^n)$.

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It is well-known that the Morrey spaces have important applications in the theory of partial differential equations, in linear as well as in non-linear theory, and the Herz spaces play an important role in characterizing the properties of functions and multipliers on the classical Hardy spaces. In 2005, Lu and Xu [3] introduced the following Morrey-Herz spaces:

DEFINITION 1.1 (See [3]). Let $\alpha \in \mathbf{R}^1$, $0 < p \leq \infty$, $0 < q < \infty$ and $\lambda \geq 0$. The homogeneous Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(R^n) = \{f \in L_{loc}^q(R^n - \{0\}) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} < \infty\}, \tag{4}$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)} = \sup_{k_0 \in Z} 2^{-k_0\lambda} \left\{ \sum_{k=-\infty}^{k_0} 2^{k\alpha p} \|f\varphi_k\|_q^p \right\}^{1/p}, \tag{5}$$

with the usual modifications made when $p = \infty$. We can similarly define the non-homogeneous Morrey-Herz spaces $MK_{p,q}^{\alpha,\lambda}(R^n)$. It is easy to see that $M\dot{K}_{p,q}^{\alpha,0}(R^n) = \dot{K}_q^{\alpha,p}(R^n)$ and $M_q^\lambda(R^n) \subset M\dot{K}_{q,q}^{0,\lambda}(R^n)$. In particular, $\dot{K}_p^{0,p}(R^n) = L^p(R^n)$, $\dot{K}_p^{(\alpha/p),p}(R^n) = L^p(|x|^\alpha dx)$.

The aim of this paper is to introduce the following new weighted Morrey-Herz spaces:

DEFINITION 1.2. Let $\alpha \in \mathbf{R}^1$, $0 < p \leq \infty$, $0 < q < \infty$, $\lambda \geq 0$ and ω_1 and ω_2 be non-negative weight functions. The homogeneous weighted Morrey-Herz space $M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1, \omega_2)$ is defined by

$$M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1, \omega_2) = \{f \in L_{loc}^q(R^n - \{0\}, \omega_2) : \|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1,\omega_2)} < \infty\}, \tag{6}$$

where

$$\|f\|_{M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1\omega_2)} = \sup_{k_0 \in Z} \omega_1(B_{k_0})^{-(\lambda/n)} \left\{ \sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p/n} \|f\varphi_k\|_{q,\omega_2}^p \right\}^{1/p}. \tag{7}$$

It is easy to see that when $\omega_1 = \omega_2 = 1$, we have $M\dot{K}_{p,q}^{\alpha,\lambda}(1,1) = M\dot{K}_{p,q}^{\alpha,\lambda}(R^n)$. We can similarly define the non-homogeneous weighted Morrey-Herz spaces $MK_{p,q}^{\alpha,\lambda}(\omega_1, \omega_2)$.

2 Some Applications

As applications, we can discuss the boundedness of many operators on the weighted Morrey-Herz spaces. They are significant generalizations of many known results. For example, the classical Hardy-Littlewood mean operator T_0 is defined by

$$T_0(f, x) = \frac{1}{x} \int_0^x f(t)dt, \ x > 0. \tag{8}$$

In 1984, Carton-Lebrun and Fosset [4] introduced the weighted Hardy-Littlewood mean operator T defined by

$$T(f,x)=\int_0^1 f(tx)\psi(t)dt,\quad x=(x_1,x_2,\ldots,x_n)\in\mathbf{R}^n, \tag{9}$$

where $tx=(tx_1,tx_2,\ldots,tx_n)$ denotes an isotropic dilation, and $\psi:[0,1]\to[0,\infty)$ is a function, f be a measurable complex valued function on $\mathbf{R}^n$. If $\psi=1$ and $n=1$, then T reduces to T_0. In what follows, A_∞ denotes the weight function class of B. Muckenhoupt, that is, there is a constant C independent of the cube Q in $\mathbf{R}^n$, such that

$$\left(\frac{1}{|Q|}\int_Q \omega(x)dx\right)\exp\left\{\frac{1}{|Q|}\int_Q \ln(\frac{1}{\omega(x)})dx\right\}\le C,\ \text{all } Q\subset\mathbf{R}^n, \tag{10}$$

where $|Q|$ is the Lebesgue measure of Q (see [5]).

In 2001 Xiao [6] obtained the $L^p(R^n)$ bounds of the operator T is defined by (9). In this section, we obtain the following results.

THEOREM 2.1. Let $\alpha\in\mathbf{R}^1$, $0<p<\infty$, $1\le q<\infty$ and $\lambda>0$. Let ψ be a real-valued nonnegative measurable function defined on $[0,1]$, and $\omega_1\in A_\infty$, a non-negative weight function ω_2 which satisfies

$$\omega_2(tx)=t^\beta\omega_2(x),\quad t>0,\quad \beta\in\mathbf{R}^1, \tag{11}$$

$\|T\|$ be the norm of the operator T which is defined by (9):

$$M\dot{K}^{\alpha,\lambda}_{p,q}(\omega_1,\omega_2)\to M\dot{K}^{\alpha,\lambda}_{p,q}(\omega_1,\omega_2).$$

(1) If $t^{-(\beta+n)/q}\psi(t)$ is a concave function on [0,1] and $\int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q}\psi(t)dt<\infty$. Then

$$\|T\|\le C(p,\alpha,\lambda)\int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q}\psi(t)dt, \tag{12}$$

where

$$C(p,\alpha,\lambda)=\begin{cases} C_0^{(\alpha-\lambda)/n}2^{(1/p)-2}(1+p)^{1/p}(1+2^{|\alpha-\lambda|\delta}),\ 0<p<1,\\ C_0^{(\alpha-\lambda)/n}2^{1-(2/p)}(1+(1/p))(1+2^{|\alpha-\lambda|\delta}),\ 1\le p<\infty,\end{cases} \tag{13}$$

(where C_0 and δ are the constants given in (21), see § 3 below).

(2) If $\|T\|<\infty$, then

$$\int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q}\psi(t)dt\le\|T\|. \tag{14}$$

REMARK 1. ω_2 is an extension of the power weight $\omega_2(x)=|x|^\beta$, $(x\in\mathbf{R}^n)$. We use the following notation

$$MKF=\{f\in M\dot{K}^{\alpha,\lambda}_{p,q}(\omega_1,\omega_2): F(t)=\sup_{x\in R^n}|f(tx)|\psi(t)\text{ is a concave function on }[0,1]\}.$$

Then MKF is a subspace of the space $M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1,\omega_2)$.

THEOREM 2.2. Let $\alpha \in \mathbf{R}^1$, $0 < p < \infty$, $0 < q < 1$ and $\lambda > 0$. Let ψ be a real-valued nonnegative measurable function defined on $[0,1]$, and ω_1, ω_2 are as in Theorem 2.1 and $\|T\|$ be the norm of the operator T defined by (9): $MKF \to M\dot{K}_{p,q}^{\alpha,\lambda}(\omega_1,\omega_2)$.

(1) If $t^{-(\beta+n)/q}\psi(t)$ is a concave function on $[0,1]$, and $\int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q}\psi(t)dt < \infty$, then

$$\|T\| \le C(p,q,\alpha,\lambda)\int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q}\psi(t)dt, \tag{15}$$

where $C(p,q,\alpha,\lambda)$ is given by

$$\begin{cases} C_0^{(\alpha-\lambda)/n}2^{(1/p)-(1/q)-2}q^{-1/p}(p+q)^{1/p}(1+q)^{1/q}(1+2^{|\lambda-\alpha)|\delta}), & 0 < p \le q < 1, \\ C_0^{(\alpha-\lambda)/n}2^{(1/q)-2}(1+q)^{1/q}(1+2^{|\lambda-\alpha|\delta}), & 0 < q \le p < 1, \\ C_0^{(\alpha-\lambda)/n}2^{(1/q)-(2/p)-1}(1+(1/p))(1+q)^{1/q}(1+2^{|\lambda-\alpha|\delta}), & 0 < q < 1 \le p < \infty, \end{cases} \tag{16}$$

where C_0 and δ are the constants given in (21), see § 3 below.

(2) If $\|T\| < \infty$, then

$$\int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q}\psi(t)dt \le \|T\|. \tag{17}$$

REMARK 2. There are some similar results for the non-homogeneous weighted Morrey-Herz spaces. We omit the details here.

REMARK 3. Take limits as $\lambda \to 0^+$ in Theorem 2.1 and 2.2, we obtain the corresponding results of the operator T is defined by (9) on the weighted Herz spaces.

3 Proofs of Theorems

We require the following Lemmas to prove our results.

LEMMA 3.1. Let f be a nonnegative measurable function on $[0,1]$. If $1 \le p < \infty$, then

$$\left(\int_0^1 f\right)^p \le \int_0^1 f^p. \tag{18}$$

Lemma 3.1 is an immediate consequences of Hölder inequality.

LEMMA 3.2 (See [7, 8]). Let f be a nonnegative measurable and concave function on $[a,b]$, $0 < \alpha \le \beta$. Then

$$\left\{\frac{\beta+1}{b-a}\int_a^b [f(x)]^\beta dx\right\}^{\frac{1}{\beta}} \le \left\{\frac{\alpha+1}{b-a}\int_a^b [f(x)]^\alpha dx\right\}^{\frac{1}{\alpha}}. \tag{19}$$

Set $a=0$, $b=1$. For $\alpha = p$, $\beta = 1$, that is, $0 < p \le 1$, we obtain from (19) that

$$\left(\int_0^1 f\right)^p \le \frac{p+1}{2^p}\int_0^1 f^p. \tag{20}$$

By the properties of A_∞ weights, we have

LEMMA 3.3 (See [5]). If $\omega \in A_\infty$, then there exist $\delta > 0$, $C_0 > 0$, such that for each ball $\mathbf{B}$ and measurable subset E of $\mathbf{B}$,

$$\frac{\omega(E)}{\omega(B)} \le C_0 \left(\frac{|E|}{|B|}\right)^\delta. \tag{21}$$

where $|E|$ is the Lebesgue measure of E and $\omega(E) = \int_E \omega(x)dx$.

LEMMA 3.4 (See [8]). (C_p inequality) Let $a_1, a_2, \ldots, a_n$ be arbitrary real (or complex) numbers, then

$$\left(\sum_{k=1}^n |a_k|\right)^p \le C_p \sum_{k=1}^n |a_k|^p,\ 0 < p < \infty, \tag{22}$$

where

$$C_p = \begin{cases} 1, & 0 < p < 1, \\ n^{p-1}, & 1 \le p < \infty. \end{cases} \tag{23}$$

In what follows, we shall write simply $M\dot{K}^{\alpha,\lambda}_{p,q}(\omega_1,\omega_2)$ to denote MK.

PROOF OF THEOREM 2.1. First, we prove (12). Using Minkowski's inequality for integrals and (11), and setting $u = tx$, we get

$$\begin{aligned} \|(Tf)\varphi_k\|_{q,\omega_2} &\le \int_0^1 \{\int_{D_k} |f(tx)|^q \omega_2(x)dx\}^{1/q}\psi(t)dt \\ &= \int_0^1 \{\int_{2^{k-1}t<|u|\le 2^k t} |f(u)|^q \omega_2(u)du\}^{1/q} t^{-(\beta+n)/q}\psi(t)dt. \end{aligned}$$

For each $t \in (0,1)$, there exists an integer m such that $2^{m-1} < t \le 2^m$. Setting

$$A_{k,m} = \{u \in R^n : 2^{k+m-1} < |u| \le 2^{k+m}\},$$

we obtain

$$\begin{aligned} \|(Tf)\varphi_k\|_{q,\omega_2} &\le \int_0^1 \{\int_{A_{(k-1),m}} |f(u)|^q \omega_2(u)du \\ &\quad + \int_{A_{k,m}} |f(u)|^q \omega_2(u)du\}^{1/q} t^{-(\beta+n)/q}\psi(t)dt \\ &\le \int_0^1 (\|f\varphi_{k+m-1}\|_{q,\omega_2} + \|f\varphi_{k+m}\|_{q,\omega_2}) t^{-(\beta+n)/q}\psi(t)dt. \end{aligned} \tag{24}$$

It follows that

$$\begin{aligned} \|Tf\|_{MK} &\le \sup_{k_0\in Z} [\omega_1(B_{k_0})]^{-(\lambda/n)} \{\sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p/n} \\ &\quad \times [\int_0^1 (\|f\varphi_{k+m-1}\|_{q,\omega_2} + \|f\varphi_{k+m}\|_{q,\omega_2}) t^{-(\beta+n)/q}\psi(t)dt]^p\}^{1/p}. \end{aligned} \tag{25}$$

Now, we consider two cases for p:

CASE 1. $0 < p < 1$. In this case, it follows from (25) and (20) that

$$\begin{aligned}\|Tf\|_{MK} &\leq \tfrac{(1+p)^{1/p}}{2} \sup_{k_0 \in Z} [\omega_1(B_{k_0})]^{-(\lambda/n)} \{ \sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p/n} \\ &\quad \times \int_0^1 (\|f\varphi_{k+m-1}\|_{q,\omega_2}^p + \|f\varphi_{k+m}\|_{q,\omega_2}^p) t^{-(\beta+n)p/q} \psi^p(t) dt \}^{1/p} \\ &\leq 2^{(1/p)-2}(1+p)^{1/p} \sup_{k_0 \in Z} [\omega_1(B_{k_0})]^{-(\lambda/n)} \\ &\quad \times \{ [\int_0^1 \sum_{k=-\infty}^{k_0} \omega_1(B_{k+m-1})^{\alpha p/n} \|f\varphi_{k+m-1}\|_{q,\omega_2}^p (\tfrac{\omega_1(B_k)}{\omega_1(B_{k+m-1})})^{\alpha p/n} \\ &\quad \times t^{-(\beta+n)p/q} \psi^p(t) dt]^{1/p} + [\int_0^1 \sum_{k=-\infty}^{k_0} \omega_1(B_{k+m})^{\alpha p/n} \|f\varphi_{k+m}\|_{q,\omega_2}^p \\ &\quad \times (\tfrac{\omega_1(B_k)}{\omega_1(B_{k+m})})^{\alpha p/n} t^{-(\beta+n)p/q} \psi^p(t) dt]^{1/p} \}.\end{aligned} \tag{26}$$

By (21) and $|B_k| = \frac{\pi^{n/2}}{\Gamma((n/2)+1)} 2^{kn}$, we have

$$\frac{\omega_1(B_k)}{\omega_1(B_{k+m-1})} \leq C_0 (\frac{|B_k|}{|B_{k+m-1}|})^{\delta} = C_0 2^{-(m-1)n\delta} \tag{27}$$

and

$$\frac{\omega_1(B_k)}{\omega_1(B_{k+m})} \leq C_0 2^{-mn\delta}. \tag{28}$$

It follows from (26), (27) and (28) that

$$\begin{aligned}\|Tf\|_{MK} &\leq C_0^{(\alpha-\lambda)/n} 2^{(1/p)-2}(1+p)^{1/p} \|f\|_{MK} \int_0^1 (2^{-(m-1)(\alpha-\lambda)\delta} \\ &\quad + 2^{-m(\alpha-\lambda)\delta}) t^{-(\beta+n)/q} \psi(t) dt \\ &\leq C_0^{(\alpha-\lambda)/n} 2^{(1/p)-2}(1+p)^{1/p} (1 + 2^{|\alpha-\lambda|\delta}) \|f\|_{MK} \\ &\quad \times \int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q} \psi(t) dt.\end{aligned} \tag{29}$$

CASE 2. $1 \leq p < \infty$. In this case, it follows from (25), (18), (20) and (28) that

$$\begin{aligned}\|Tf\|_{MK} &\leq 2^{1-(1/p)} \sup_{k_0 \in Z} [\omega_1(B_{k_0})]^{-(\lambda/n)} \{ \textstyle\sum_{k=-\infty}^{k_0} [\omega_1(B_k)]^{\alpha p/n} \\ &\quad \times \int_o^1 (\|f\varphi_{k+m-1}\|_{q,\omega_2}^p + \|f\varphi_{k+m}\|_{q,\omega_2}^p) t^{-(\beta+n)p/q} \psi^p(t) dt \}^{1/p} \\ &\leq C_0^{(\alpha-\lambda)/n} 2^{1-(2/p)} (1 + (1/p))(1 + 2^{|\alpha-\lambda|\delta}) \|f\|_{MK} \\ &\quad \times \int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q)} \psi(t) dt.\end{aligned} \tag{30}$$

Hence, by (29) and (30), we get

$$\|T\| \leq C(p, \alpha, \lambda) \int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q} \psi(t) dt, \tag{31}$$

where $C(p, \alpha, \lambda)$ is defined by (13).

To prove the opposite inequality, putting $\varepsilon \in (0, 1)$, we set $\omega_1(B_k) = 2^{kn\delta}$, $\omega_2(x) = |x|^{\beta}$ and

$$f_0(x) = |x|^{(\lambda-\alpha)\delta-(\beta+n)/q}, \ x \in \mathbf{R}^n. \tag{32}$$

We need to consider two cases :

CASE 1. $\alpha \neq \lambda$. Then

$$\|f_0\varphi_k\|_{q,\omega_2}^q = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{2^{k-1}}^{2^k} r^{(\lambda-\alpha)q\delta-(\beta+n)} r^{n-1} r^{\beta} dr = C_n 2^{k(\lambda-\alpha)q\delta},$$

where

$$C_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \left| \frac{1-2^{-(\lambda-\alpha)q\delta}}{(\lambda-\alpha)q\delta} \right|.$$

It follows that

$$\|f_0\|_{MK} = \sup_{k_0 \in Z} 2^{-k_0\lambda\delta} \{ \sum_{k=-\infty}^{k_0} 2^{kp\alpha\delta} (C_n^{1/q} 2^{k(\lambda-\alpha)\delta})^p \}^{1/p} = C_n^{1/q} \frac{1}{(2^{p\lambda\delta}-1)^{1/p}}. \quad (33)$$

CASE 2. $\alpha = \lambda$. Then $\|f_0\varphi_k\|_{q,\omega_2}^q = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_{2^{k-1}}^{2^k} r^{-1} dr = \frac{2\pi^{n/2}}{\Gamma(n/2)} \ln 2$. Thus

$$\|f_0\|_{MK} = (\frac{2\pi^{n/2} \ln 2}{\Gamma(n/2)})^{1/q} \times (2^{p\lambda\delta} - 1)^{-(1/p)}. \quad (34)$$

It follows from (33) and (34) that $f_0 \in MK$. By (9), we obtain

$$T(f_0, x) = \int_0^1 f_0(tx)\psi(t)dt = f_0(x) \int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q} \psi(t)dt. \quad (35)$$

and $\|Tf_0\|_{MK} = \|f_0\|_{MK} \int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q} \psi(t)dt$. Thus,

$$\|T\| \geq \frac{\|Tf_0\|_{MK}}{\|f_0\|_{MK}} = \int_0^1 t^{(\lambda-\alpha)\delta-(\beta+n)/q} \psi(t)dt. \quad (36)$$

This completes the proof of Theorem 2.1.

The idea of proof of theorem 2.2 is similar to that of Theorem 2.1, we omit the details here.

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