Asymptotic Bounds For Solutions Of A Periodic Reaction Diffusion System

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Abstract

The purpose of this paper is to investigate the asymptotic behavior of non-negative solutions of a periodic reaction diffusion system. By De Giorgi iteration technique, we obtain the a priori upper bound of non-negative periodic solutions of the considered periodic system. We then establish the existence of the maximum periodic solution and asymptotic bounds of non-negative solutions of the initial boundary value problem.

1 Introduction

In this paper, we consider the following periodic reaction diffusion system

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \text{div}(\nabla |\nabla u|^p - 2 \nabla u) + b_1 u^{\alpha_1} v^{\beta_1}, \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad (1) \\
\frac{\partial v}{\partial t} &= \text{div}(\nabla |\nabla v|^p - 2 \nabla u) + b_2 u^{\alpha_2} v^{\beta_2}, \quad (x, t) \in \Omega \times \mathbb{R}^+, \quad (2) \\
u(x, t) &= v(x, t) = 0, \quad (x, t) \in \partial \Omega \times \mathbb{R}^+, \quad (3) \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega, \quad (4)
\end{align*}
\]

where \( p_1, p_2 > 2, \alpha_1, \alpha_2, \beta_1, \beta_2 \geq 1, \alpha_1 < p_1 - 1, \beta_1 < p_1 - \alpha_1 - 1, \alpha_2 < p_2 - \beta_2 - 1, \beta_2 < p_2 - 1, \Omega \subset \mathbb{R}^n \) is a bounded domain with smooth boundary, \( b_1 = b_1(x, t) \) and \( b_2 = b_2(x, t) \) are nonnegative continuous functions and \( T \)-periodic (\( T > 0 \)) with respect to \( t \), and \( u_0, v_0 \) are nonnegative bounded smooth functions.

In recent years, periodic reaction diffusion equations and systems are of particular interests since they can take into account periodic fluctuations occurring in the phenomena appearing in the models, and have been extensively studied by many researchers (see e.g. [1]-[5]). The models for the evolution of the biological species living in the periodic environment are often described by coupled systems of periodic nonlinear diffusion equations, and therefore it is important to study the existence and asymptotic behavior of solutions of these systems. To our knowledge, however, it seems that there are few papers that deal with the asymptotic behavior of periodic-parabolic systems with degeneracy.

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This work is an extension of [6]. We establish the existence of nontrivial nonnegative periodic solutions of the problem (1)-(4) and their asymptotic behavior. Since (1) and (2) have periodic sources, it is not appropriate to consider the steady state approach and we shall seek some new approaches. Our idea is to consider all nonnegative periodic solutions, which will be showed to have a priori upper bound $C_0$ according to the maximum norm. Then by monotonicity method we show the existence of the maximum periodic solution and asymptotic bounds of nonnegative solutions of the initial boundary value problem.

2 Preliminaries

Since (1) and (2) are degenerate whenever $|\nabla u| = |\nabla v| = 0$, we focus our main efforts on the discussion of weak solutions.

DEFINITION 1. A vector-valued function $(u, v)$ is said to be a weak upper-solution to the problem (1)-(4) in $Q_\tau = \Omega \times (0, \tau)$ with $\tau > 0$, if

$$u \in L^{p_1}(0, \tau; W^{1,p_1}(\Omega)) \cap L^\infty(Q_\tau), \quad v \in L^{p_2}(0, \tau; W^{1,p_2}(\Omega)) \cap L^\infty(Q_\tau),$$

and for any nonnegative function $\varphi \in C^1(\overline{Q}_\tau)$ with $\varphi|_{\partial \Omega \times (0, \tau)} = 0$, we have

$$\left\{ \begin{align*}
\int_\Omega u(x, \tau)\varphi(x, \tau)dx - \int_\Omega u_0(x)\varphi(x, 0)dx - \int_{Q_\tau} u \frac{\partial \varphi}{\partial t} dxdt \\
+ \int_{Q_\tau} |\nabla u|^{p_1-2}\nabla u \nabla \varphi dxdt \geq \int_{Q_\tau} b_1 u^{a_1}v^{b_1} \varphi dxdt,
\end{align*} \right.$$

$$\left\{ \begin{align*}
\int_\Omega v(x, \tau)\varphi(x, \tau)dx - \int_\Omega v_0(x)\varphi(x, 0)dx - \int_{Q_\tau} v \frac{\partial \varphi}{\partial t} dxdt \\
+ \int_{Q_\tau} |\nabla v|^{p_2-2}\nabla v \nabla \varphi dxdt \geq \int_{Q_\tau} b_2 u^{a_2}v^{b_2} \varphi dxdt,
\end{align*} \right.$$

$$u(x, t) \geq 0, \quad v(x, t) \geq 0, \quad (x, t) \in \partial \Omega \times (0, \tau),$$

$$u(x, 0) \geq u_0(x), \quad v(x, 0) \geq v_0(x), \quad x \in \Omega.$$

Replacing “$\geq$” by “$\leq$” in the above inequalities, it follows the definition of a weak lower-solution. Furthermore, if $(u, v)$ is a weak upper-solution as well as a weak lower-solution, then we call it a weak solution of the problem (1)-(4).

DEFINITION 2. A vector-valued function $(u, v)$ is said to be a $T$-periodic solution of the problem (1)-(3), if it is a solution such that $u(\cdot, 0) = u(\cdot, T)$, $v(\cdot, 0) = v(\cdot, T)$ in $\Omega$. A vector-valued function $(\overline{u}, \overline{v})$ is said to be a $T$-periodic upper-solution of the problem (1)-(3), if it is an upper-solution such that $\overline{u}(\cdot, 0) \geq \overline{u}(\cdot, T)$, $\overline{v}(\cdot, 0) \geq \overline{v}(\cdot, T)$ in $\Omega$. A vector-valued function $(\underline{u}, \underline{v})$ is said to be a $T$-periodic lower-solution of the problem (1)-(3), if it is a lower-solution such that $\underline{u}(\cdot, 0) \leq \underline{u}(\cdot, T)$, $\underline{v}(\cdot, 0) \leq \underline{v}(\cdot, T)$ in $\Omega$. A pair of $T$-periodic upper-solution $(\overline{u}, \overline{v})$ and $T$-periodic lower-solution $(\underline{u}, \underline{v})$ are said to be ordered if $\overline{u} \geq \underline{u}$, $\overline{v} \geq \underline{v}$ in $Q_T = \Omega \times (0, T)$.

LEMMA 1 ([6]). Let $(\underline{u}, \underline{v})$ be a lower-solution of the problem (1)-(4) with the initial value $(\underline{u}_0, \underline{v}_0)$, and $(\overline{u}, \overline{v})$ an upper-solution of the problem (1)-(4) with the initial value $(\overline{u}_0, \overline{v}_0)$. Then $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$ a.e. in $Q_T$ if $\underline{u}_0 \leq \overline{u}_0$, $\underline{v}_0 \leq \overline{v}_0$ a.e. in $\Omega$. 

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LEMMA 2. (see [6]) For any nonnegative bounded initial value, the problem (1)-(4) admits a global nonnegative solution, and the problem (1)-(3) admits a nontrivial nonnegative periodic solution.

The main results of this paper is the following theorem.

THEOREM 1. The problem (1)-(3) admits a maximal periodic solution \((U, V)\).

Moreover, if \((u, v)\) is the solution of the initial boundary value problem (1)-(4) with nonnegative initial value \((u_0, v_0)\), then for any \(\varepsilon > 0\), there exists \(t_1\) depending on \(u_0\) and \(\varepsilon\), \(t_2\) depending on \(v_0\) and \(\varepsilon\), such that

\[
0 \leq u \leq U + \varepsilon, \quad \text{for } x \in \Omega, \quad t \geq t_1,
\]

\[
0 \leq v \leq V + \varepsilon, \quad \text{for } x \in \Omega, \quad t \geq t_2.
\]

3 Proofs

In this section, we prove the main results of this paper. Firstly, we establish some important estimates on nonnegative periodic solutions of the problem (1)-(3).

LEMMA 3. Let \((u, v)\) be a nonnegative periodic solution of the problem (1)-(3). Then there exists positive constants \(r\) and \(s\) such that

\[
\alpha_2 \frac{p_1 + r - 1}{p_2 - \beta_2 - 1} < \frac{p_1 - \alpha_1 - 1}{\beta_1} < \frac{p_1 + r - 1}{p_2 - s - 1} < \frac{p_1 - \alpha_1 - 1}{\beta_1}
\]

and

\[
\|u\|_{L^r(\Omega_T)} \leq C, \quad \|v\|_{L^s(\Omega_T)} \leq C.
\]

In addition, we have

\[
\iint_{Q_T} |\nabla u|^{p_1} dx dt \leq C, \quad \iint_{Q_T} |\nabla v|^{p_2} dx dt \leq C,
\]

where \(C\) is a positive constant depending on \(p_1, p_2, \alpha_1, \beta_2, r, s\) and \(|\Omega|\).

PROOF. For \(r > 1\), multiplying (1) by \(u^{r-1}\) and integrating over \(\Omega_T\), we deduce

\[
\iint_{Q_T} \frac{\partial u}{\partial t} u^{r-1} dx dt + \iint_{Q_T} |\nabla u|^{p_1-2} \nabla u \nabla u^{r-1} dx dt = \iint_{Q_T} b_1 u^{\alpha_1 + r - 1} v^{\beta_1} dx dt.
\]

By the periodic boundary value condition, we see that the first term of the left hand side of the above equality vanishes. That is

\[
(r - 1) \left(\frac{p_1}{p_1 + r - 2}\right)^{p_1} \iint_{Q_T} |\nabla u|^{\frac{p_1 + r - 2}{p_1}} dx dt = \iint_{Q_T} b_1 u^{\alpha_1 + r - 1} v^{\beta_1} dx dt.
\]

Then we have

\[
\iint_{Q_T} |\nabla u|^{\frac{p_1 + r - 2}{p_1}}^{p_1} dx dt \leq B_1 \left(\frac{p_1 + r - 2}{p_1}\right)^{p_1} \iint_{Q_T} u^{\alpha_1 + r - 1} v^{\beta_1} dx dt,
\]
From (7) we have
\[ \int \int_{Q_T} u^{p_1 + r - 2} \, dx \, dt \leq C_{p_1} \int \int_{Q_T} |\nabla u|^{p_1} \, dx \, dt \]
\[ \leq \frac{C_{p_1} B_1}{r - 1} \left( \frac{p_1 + r - 2}{p_1} \right)^{p_1} \int \int_{Q_T} u^{\alpha_1 r - 1} v^{\beta_1} \, dx \, dt, \]
where \( C_{p_1} \) is a constant depending only on \(|\Omega|\) and \( N \), and it becomes very large when the measure of the domain \( \Omega \) becomes small. Notice that \( \beta_1 < p_1 - \alpha_1 - 1 \) implies \( \alpha_1 < p_1 - 1 \), then \( \alpha_1 + r - 1 < p_1 + r - 2 \). According to Young’s inequality, we have
\[ u^{\alpha_1 + r - 1} v^{\beta_1} \leq \varepsilon_1 u^{p_1 + r - 2} + C(\varepsilon_1) v^{\frac{\beta_1 (p_1 + r - 2)}{p_1 - \alpha_1 - 1}}, \]
where \( \varepsilon_1 > 0 \) and \( C(\varepsilon_1) \) are constants of Young’s inequality. Take
\[ \varepsilon_1 = \frac{1}{2} \frac{r - 1}{C_{p_1} B_1} \left( \frac{p_1}{p_1 + r - 2} \right)^{p_1}, \]
from (7) we have
\[ \int \int_{Q_T} u^{p_1 + r - 2} \, dx \, dt \leq \frac{1}{2} \int \int_{Q_T} u^{p_1 + r - 2} \, dx \, dt + C_1 \int \int_{Q_T} v^{\frac{\beta_1 (p_1 + r - 2)}{p_1 - \alpha_1 - 1}} \, dx \, dt. \]
That is
\[ \int \int_{Q_T} u^{p_1 + r - 2} \, dx \, dt \leq \frac{1}{2} \int \int_{Q_T} u^{p_1 + r - 2} \, dx \, dt + C_1 \int \int_{Q_T} v^{\frac{\beta_1 (p_1 + r - 2)}{p_1 - \alpha_1 - 1}} \, dx \, dt. \]
Also, we can get an similar estimate on \( v^s \) for \( s > 1 \), and hence
\[ \int \int_{Q_T} u^{p_1 + r - 2} \, dx \, dt + \int \int_{Q_T} v^{p_2 + s - 2} \, dx \, dt \leq C \int \int_{Q_T} u^{\frac{\beta_1 (p_1 + r - 2)}{p_1 - \alpha_1 - 1}} \, dx \, dt + C \int \int_{Q_T} v^{\frac{\alpha_2 (p_2 + s - 2)}{p_2 - \beta_2 - 1}} \, dx \, dt. \]
Since \( \beta_1 < p_1 - \alpha_1 - 1 \), \( \alpha_2 < p_2 - \beta_2 - 1 \), there must exist \( r \geq \max\{2(\alpha_1 + 1), 2\alpha_2\} \) and \( s \geq \max\{2(\beta_2 + 2), 2\beta_1\} \) such that
\[ \frac{\beta_1}{p_1 - \alpha_1 - 1} < \frac{p_2 + s - 2}{p_1 + r - 2} < \frac{p_2 - \beta_2 - 1}{\alpha_2}. \]
By Young’s inequality, we have
\[ \int \int_{Q_T} u^{\frac{\alpha_2 (p_2 + s - 2)}{p_2 - \beta_2 - 1}} \, dx \, dt \leq \varepsilon_2 \int \int_{Q_T} u^{p_1 + r - 2} \, dx \, dt + C(\varepsilon_2) |Q_T|, \]
\[ \int \int_{Q_T} v^{\frac{\alpha_2 (p_2 + s - 2)}{p_2 - \beta_2 - 1}} \, dx \, dt \leq \varepsilon_3 \int \int_{Q_T} v^{p_2 + s - 2} \, dx \, dt + C(\varepsilon_3) |Q_T|. \]
Take \( \varepsilon_2 = \frac{1}{2\varepsilon_2} \), \( \varepsilon_3 = \frac{1}{2\varepsilon_3} \), it follows from (9) that
\[
\iint_{Q_T} u^{p_1+r-2} dx dt + \iint_{Q_T} v^{p_2+s-2} dx dt \leq C.
\]
Thus we complete the proof of inequality (5).

Now we show the proof of (6). Multiplying (1) by \( u \) and integrating over \( Q_T \), by the periodic boundary value condition and Hölder’s inequality, we have
\[
\int_{Q_T} |\nabla u|^{p_1} dx dt \leq C \int_{Q_T} u^{\alpha_1+1} v^{\beta_1} dx dt
\]
\[
\leq C \left( \int_{Q_T} u^{2(\alpha_1+1)} dx dt \right)^{1/2} \left( \int_{Q_T} v^{2\beta_1} dx dt \right)^{1/2}.
\]
Due to \( r \geq \max \{2(\alpha_1+1), 2\beta_2\} \), \( s \geq \max \{2(\beta_2+1), 2\alpha_1\} \), the first inequality in (6) follows from (5) immediately. The same is true for the second inequality. The proof is completed.

In the following we show the uniform upper bound of the maximum norm of nonnegative periodic solutions.

**Lemma 4.** Let \((u, v)\) be a nonnegative periodic solution of (1)-(3). Then there is a positive constant \( C_0 \) such that
\[
\|u\|_{L^\infty(Q_T)} \leq C_0, \quad \|v\|_{L^\infty(Q_T)} \leq C_0.
\]  

**Proof.** Denote \( s_+ = \max \{s, 0\} \) and take \( k \) be a determined positive constant. Multiplying (1) by \((u - k)_+\) and integrating over \( Q_T \), we have
\[
\frac{1}{2} \iint_{Q_T} \frac{\partial}{\partial t} (u - k)_+^2 dx dt + \iint_{Q_T} |\nabla (u - k)_+|^{p_1} dx dt
\]
\[
\leq C \iint_{Q_T} u^{\alpha_1} v^{\beta_1} (u - k)_+ dx dt.
\]  

Denote \( \mu(k) = \text{mes}\{(x, t) \in Q_T : u(x, t) > k\} \). Combine Lemma 3 (with \( r \) and \( s \) large enough) with Young’s and Hölder’s inequalities, we have
\[
\frac{1}{2} \iint_{Q_T} \frac{\partial}{\partial k} (u - k)_+^2 dx dt + \iint_{Q_T} |\nabla (u - k)_+|^{p_1} dx dt
\]
\[
\leq C \left( \iint_{Q_T} (u^{\alpha_1} v^{\beta_1})^{\xi} dx dt \right)^{\frac{1}{\xi}} \left( \iint_{Q_T} (u - k)^{\xi} dx dt \right)^{1/\xi}
\]
\[
\leq C \left( \iint_{Q_T} (u - k)_+^{\xi \eta} dx dt \right)^{1/\xi \eta} \mu(k)^{(1-\frac{s}{p_1}) \xi},
\]
where constants \( \xi, \eta > 1 \) are to be determined. By Nirenberg-Gagliardo’s inequality and Lemma 3, we have
\[
\left( \iint_{Q_T} (u - k)_+^{\xi \eta} dx dt \right)^{1/\xi \eta} \leq C \left( \iint_{Q_T} |\nabla (u - k)_+|^{p_1} dx dt \right)^{\theta/p_1},
\]
where
\[ \theta = \left(1 - \frac{1}{\xi \eta}\right)\left(\frac{1}{N} - \frac{1}{p_1} + 1\right)^{-1}. \]
Combining (11) with (12) and (13), we have
\[ \int \int_{Q_T} |\nabla (u - k)|^{p_1} dx dt \leq C \left( \int \int_{Q_T} |\nabla (u - k)|^{p_1} dx dt \right)^{\frac{\mu}{p_1}} \mu(k)^{\frac{1}{1-\frac{\xi}{\eta}}}. \]  
(14)
Set
\[ w(k) = \int \int_{Q_T} |\nabla (u - k)|^{p_1} dx dt, \]
from (14) we obtain
\[ w(k) \leq C \mu(k)^{\frac{\mu}{p_1}} \mu(k)^{\frac{1}{1-\frac{\xi}{\eta}}}. \]  
(15)
Take \( k_h = M(2 - 2^{-h}), h = 0, 1, \ldots, \) and \( M > 0 \) to be determined. It follows from (13) that
\[ (k_{h+1} - k_h)^{\xi \eta} \mu(k_{h+1}) \leq \int \int_{Q_T} (u - k_h)^{\xi \eta} dx dt \leq Cw(k_h)^{\frac{\mu}{p_1}}. \]
Combining the above inequality with (15) we obtain
\[ \mu(k_{h+1}) \leq C 4^{\xi \eta} \mu(k_h)^{\frac{\mu}{p_1}} = C b^\gamma \mu(k_h)^{\gamma}, \]
where \( b = 4^{\xi \eta} \) and \( \gamma = \frac{(\eta - 1)(\xi - 1)N}{\xi \eta N(p_1 - 2) + \xi p_1 + N} \). For any \( \xi > 1 \), take \( \eta \) be a positive constant satisfying
\[ \eta > \max\{p_1, \frac{\xi p_1 + N}{\xi N} + p_1 - 1\}, \]
then we have \( \gamma > 1 \). By Lemma 3, we can select \( M \) large enough such that
\[ \mu(k_0) = \mu(M) \leq C^{-\frac{1}{\gamma + 1}} b^{-\frac{1}{\gamma + 1} N}. \]
According to Lemma 5.6 in [7, p. 95], we have \( \mu(k_h) \to 0 \), as \( h \to +\infty \), which implies \( u(x, t) \leq 2M \) in \( Q_T \).

Similarly, we can get the uniform upper bound estimate for \( \|v\|_{L^\infty(Q_T)} \). The proof is completed.

**PROOF OF THEOREM 1.** Firstly, we establish the existence of the maximal periodic solution of the periodic boundary value problem. Define a Poincaré map
\[ P = (P_1, P_2) : C(\Omega) \times C(\Omega) \to C(\Omega) \times C(\Omega) \] with \( P(u_0(x), v_0(x)) = (u(x, T), v(x, T)) \), where \( (u(x, t), v(x, t)) \) is the solution of the initial boundary value problem (1)-(4) with initial value \( (u_0(x), v_0(x)) \). A similar argument as [6] shows that the map \( P \) is well defined.

Let \( \lambda_i (i = 1, 2) \) be the first eigenvalue of the problem
\[
\begin{cases}
-\text{div}(|\nabla \varphi|^{p_i-2}\nabla \varphi) = \mu |\varphi|^{p_i-2} \varphi, & x \in \Omega' \\
\varphi = 0, & x \in \partial \Omega',
\end{cases}
\]
and $\varphi_i$ be the corresponding positive eigenfunction, where $\Omega' \supset \Omega$. It is easy to see if $x \in \Omega'$, then $\varphi_1(x) > 0$ and $\varphi_2(x) > 0$, that is $\min_{\Omega'} \varphi_1(x) > 0$ and $\min_{\Omega'} \varphi_2(x) > 0$. Let $(u_n(x,t), v_n(x,t))$ be the solution of the problem (1)-(4) with initial value

$$(u_0(x), v_0(x)) = (\overline{u}(x), \overline{v}(x)) = (K_1 \varphi_1(x), K_2 \varphi_2(x)),$$

where $K_1, K_2 > 0$ are taken as [6] such that $(\overline{u}(x), \overline{v}(x))$ be a $T$-periodic upper-solution. Then we have $(u_n(x,T), v_n(x,T)) = P^n(\overline{u}(x), \overline{v}(x))$ and

$$u_{n+1}(x,t) \leq u_n(x,t) \leq \overline{u}(x), \quad v_{n+1}(x,t) \leq v_n(x,t) \leq \overline{v}(x),$$

by comparison principle. By a rather standard argument, we conclude that there exist $u^*(x), v^*(x) \in C(\overline{\Omega})$ and a subsequence of $\{P^n(\overline{u}(x), \overline{v}(x))\}$, denoted by itself for simplicity, such that

$$(u^*(x), v^*(x)) = \lim_{n \to \infty} P^n(\overline{u}(x), \overline{v}(x)).$$

Similar to the proof in [6], we can prove that $(U(x,t), V(x,t))$, which is the even extension of the solution of the initial boundary value problem (1)-(4) with initial value $(u^*(x), v^*(x))$, is a periodic solution of (1)-(3). Moreover, by Lemma 4, we see that any nonnegative periodic solution $(u(x,t), v(x,t))$ of (1)-(3) must satisfy

$$\|u(x,t)\|_{L^\infty(Q_T)} \leq C_0, \quad \|v(x,t)\|_{L^\infty(Q_T)} \leq C_0.$$

Therefore, if we take $K_1, K_2$ also satisfy

$$K_1 \geq \frac{C_0}{\min_{x \in \Omega} \varphi_1(x)}, \quad K_2 \geq \frac{C_0}{\min_{x \in \Omega} \varphi_2(x)},$$

by the comparison principle and $u^*(x) \geq u(x,0), v^*(x) \geq v(x,0)$, we obtain $U(x,t) \geq u(x,t), V(x,t) \geq v(x,t)$, which means that $(U(x,t), V(x,t))$ is the maximal periodic solution of (1)-(3).

Let $(u(x,t), v(x,t))$ be the solution of the initial boundary problem (1)-(4) with given nonnegative initial value $(u_0(x), v_0(x)), (\omega_1(x,t), \omega_2(x,t))$ be the solution of (1)-(4) with initial value $(\omega_1(x,0), \omega_2(x,0)) = (K_1 \varphi_1(x), K_2 \varphi_2(x))$, where $R_1, R_2$ are positive constants satisfying the same conditions as $K_1, K_2$ and also

$$R_1 \geq \frac{\|u_0\|_{L^\infty}}{\min_{x \in \Omega} \varphi_1(x)}, \quad R_2 \geq \frac{\|v_0\|_{L^\infty}}{\min_{x \in \Omega} \varphi_2(x)}.$$

Then we have

$$u(x,t+kT) \leq w_1(x,t+kT), \quad v(x,t+kT) \leq w_2(x,t+kT)$$

for any $(x,t) \in Q_T, k = 0, 1, 2, \ldots$ A similar argument as [1] shows that

$$(\omega_1^*(x,t), \omega_2^*(x,t)) = (\lim_{k \to \infty} \omega_1(x,t+kT), \lim_{k \to \infty} \omega_2(x,t+kT))$$
exists and \((\omega_1^*(x, t), \omega_2^*(x, t))\) is a nontrivial nonnegative periodic solution of (1)-(3). Therefore, for any \(\varepsilon > 0\), there exists \(k_0\) such that

\[
\begin{align*}
    u(x, t + kT) &\leq \omega_1^*(x, t) + \varepsilon \leq U(x, t) + \varepsilon, \\
v(x, t + kT) &\leq \omega_2^*(x, t) + \varepsilon \leq V(x, t) + \varepsilon
\end{align*}
\]

for \(k \geq k_0\) and \((x, t) \in \overline{Q_T}\). Provided that the periodicities of \(\omega_1^*(x, t), \omega_2^*(x, t), U(x, t)\) and \(V(x, t)\) are taken into account, then the conclusion follows immediately.

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