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Solvability Of A Second Order Boundary Value Problem On An Unbounded Domain*

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Abstract

In this paper, we consider a second order nonlinear differential equation $x''(t) = f(t, x(t), x'(t))$ satisfying the boundary conditions $x(0) = x(\eta)$ and $\lim_{t\to+\infty} x(t) = 0$ where $f : [0, +\infty) \times R^2 \to R$ and $\eta > 0$. By the Leray-Schauder continuation theorem, we obtain the existence of at least one solution to the boundary value problems above. As an application, an example is also given.

1 Introduction

In this paper, we consider the second order nonlinear differential equation

$$x''(t) = f(t, x(t), x'(t)) \tag{1}$$

satisfying the boundary conditions

$$x(0) = x(\eta), \quad \lim_{t\to+\infty} x(t) = 0, \tag{2}$$

where $f : [0, \infty) \times R^2 \to R$ and η is a positive constant.

In recent years, multipoint boundary value problems for second-order differential equations have been widely studied. Meanwhile, boundary value problems in an infinite interval arose in many applications and received much attention. Ma [1] studied the second order boundary value problem

$$\begin{cases} y''(t) + f(t, y(t), y'(t)) = 0, & a.e. \text{ in } (0, \infty), \\ y(0) = 0, & y \text{ bounded on } [0, \infty) \end{cases} \tag{3}$$

and showed the existence of positive solutions. In [2], H. Lian and W. Ge considered the boundary value problem

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, & 0 < t < \infty, \\ x(0) = \alpha x(\eta), & \lim_{t\to+\infty} x'(t) = 0, \end{cases} \tag{4}$$

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where $\alpha \neq 1$ and $\eta > 0$. The existence of at least one solution was considered by the Leray-Schauder continuation theorem. While $\alpha = 1$, the authors [8] considered the solvability of (4) by Mawhin continuation theorem. In [3], Nickolai Kosmatov dealt with the second order nonlinear differential equation

$$(p(t)u'(t))' = f(t, u(t), u'(t)), \quad a.e.\ \ in\ \ (0, \infty)$$

satisfying two sets of boundary conditions:

$$u'(0) = 0,\ \lim_{t\to\infty} u(t) = 0 \text{ and } u(0) = 0,\ \lim_{t\to\infty} u(t) = 0.$$

Motivated by the work mentioned above, we aim to discuss the solvability of (1),(2). In general, by integrating both sides of an equation and using the boundary value conditions, one can obtain the expression of the solution. Then, by means of some suitable fixed point theorem, the sufficient conditions of the existence of solutions for the corresponding BVP can be obtained. However, (1),(2) are special and it is not easy to discuss the solvability of (1),(2) directly. So we cannot follow the conventional routine. By varying (1) appropriately, we obtain some sufficient conditions for the solvability of (1),(2).

The organization of the paper is as follows. In section 1, we introduce several recent results in the theory of boundary value problems on unbounded domains. In section 2, the background definitions and some statements are introduced. Section 3 contains the main results of this paper. At last, an example is provided to illustrate our results.

2 Preliminaries and Lemmas

In this section, we present some definitions and lemmas that will be used in this paper.

DEFINITION 2.1. A mapping defined on a Banach space is completely continuous if it is continuous and maps bounded sets into relatively compact sets.

DEFINITION 2.2. The map $f : [0, \infty) \times R^n \to R$, $(t, z) \mapsto f(t, z)$ is $L^1[0, +\infty)$-Carathéodory, if the following conditions are satisfied:

(1) for each $z \in R^n$, the mapping $t \mapsto f(t, z)$ is Lebesgue measurable;

(2) for a.e. $t \in [0, \infty)$, the mapping $z \mapsto f(t, z)$ is continuous on R^n;

(3) for each $r > 0$, there exists an $\alpha_r \in L^1[0, \infty)$ such that, for a.e. $t \in [0, \infty)$ and every z such that $|z| \leq r$, we have $|f(t, z)| \leq \alpha_r(t)$.

LEMMA 2.1. Let X be the space of all bounded continuous vector-valued functions on $[0, \infty)$ and $S \subset X$. Then S is relatively compact in X if the following conditions hold:

(1) S is bounded in X;

(2) the functions from S are equicontinuous on any compact interval of $[0, \infty)$;

(3) the functions from S are equiconvergent, that is , given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $\|\phi(t) - \phi(\infty)\| < \varepsilon$, for all $t > T$ and all $\phi \in S$.

Let

$$X = \{x \in C^1[0, +\infty) : x(0) = x(\eta),\ \lim_{t\to+\infty} x(t) \text{ exists},\ \lim_{t\to+\infty} x'(t) \text{ exists}\}$$

with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty\}$, where $\|x\|_\infty = \sup_{t\in[0,+\infty)} |x(t)|$ and the Lebesgue space $Z = L^1[0,+\infty)$ with the usual norm denoted by $\|x\|_1 = \int_0^\infty |x(t)|\, dt$. It is easy to show that $(X, \ \|\cdot\|)$ is a Banach space.

For the sake of studying (1),(2), we first consider the following BVP:

$$\begin{cases} x''(t) - M^2 x = \sigma(t) \\ x(0) = x(\eta), \quad \lim\limits_{t\to+\infty} x(t) = 0, \end{cases} \tag{5}$$

where M is a positive constant such that $e^{-Mt}\sigma(t) \in L^1[0,\infty)$.

LEMMA 2.2. $x(t)$ is a solution of (5) if and only if $x(t) \in X$ is a solution of the following integral equation

$$x(t) = \int_0^{+\infty} G(t,s)\sigma(s)ds \tag{6}$$

where

$$G(t,s) = \begin{cases} \frac{1}{2M(1-e^{-M\eta})}(e^{-M(t+s)} - e^{-M(t-s)}), \\ \quad 0 \le s \le \min\{t,\eta\} < \infty; \\ -\frac{1}{2M}e^{M(t-s)} + \frac{1}{2M(1-e^{-M\eta})}e^{-M(t+s)} - \frac{e^{-M\eta}}{2M(1-e^{-M\eta})}e^{-M(t-s)}, \\ \quad 0 \le t \le s \le \eta < \infty; \\ -\frac{1}{2M}e^{M(t-s)} + \frac{1-e^{M\eta}}{2M(1-e^{-M\eta})}e^{-M(t+s)}, \\ \quad 0 \le \max\{t,\eta\} \le s < \infty; \\ \frac{1-e^{M\eta}}{2M(1-e^{-M\eta})}e^{-M(t+s)} - \frac{1}{2M}e^{-M(t-s)}, \\ \quad 0 \le \eta \le s \le t < \infty. \end{cases} \tag{7}$$

PROOF. If x is a solution of problem (5), then it is easy to know that the general solution for the equation in boundary value problem (5) is as follows:

$$x(t) = Ae^{Mt} + Be^{-Mt} + \frac{1}{2M}\int_0^t [e^{M(t-s)} - e^{-M(t-s)}]\sigma(s)ds$$

where A, B are constants. Since $x(t)$ should satisfy the boundary condition (2), we get $A = -\frac{1}{2M}\int_0^\infty e^{-Ms}\sigma(s)ds$. At the same time,

$$B = \frac{1-e^{M\eta}}{2M(1-e^{-M\eta})}\int_0^\infty e^{-Ms}\sigma(s)ds + \frac{1}{2M(1-e^{-M\eta})}\int_0^\eta [e^{M(\eta-s)} - e^{-M(\eta-s)}]\sigma(s)ds.$$

Substituting the expressions of A, B into the expression of $x(t)$, after tedious computation, we get the result.

Conversely, if $x(t) \in X$ is a solution of (6), it is easy to obtain x satisfies (5).

From the expression of $G(t,s)$, we can easily get

$$|G(t,s)| \le \frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1-e^{-M\eta})}, \quad \left|\frac{\partial G(t,s)}{\partial t}\right| \le \frac{3 + e^{M\eta} - e^{-M\eta}}{2(1-e^{-M\eta})}$$

for all $t, s \in [0, \infty)$.

Consider the following BVP

$$\begin{cases} x''(t) - M^2 x = -M^2 u + f(t, u(t), u'(t)) \\ x(0) = x(\eta), \quad \lim_{t \to +\infty} x(t) = 0. \end{cases} \tag{8}$$

From Lemma 2.2, we can obtain that the solution of (8) can be expressed as

$$x(t) = \int_0^{+\infty} G(t, s)(-M^2 u + f(t, u(s), u'(s)))ds. \tag{9}$$

It is obvious that (1) is equal to $x'' - M^2 x = -M^2 x + f(t, x(t), x'(t))$. Then, for all $x \in X$, we consider

$$\begin{cases} x''(t) - M^2 x = -M^2 x + f(t, x(t), x'(t)) \\ x(0) = x(\eta), \quad \lim_{t \to +\infty} x(t) = 0. \end{cases} \tag{10}$$

Then, $x \in X$ is a solution of BVP(1),(2) if and only if $x(t) \in X$ is a solution of BVP(10).

In the following, we assume

(A_1) There exist a positive constant M such that $-M^2 x + f(t, x, x')$ is $L^1[0, +\infty)$-Carath$\acute{e}$odory and $T : X \to X$ is an operator defined by

$$Tx(t) = \int_0^{\infty} G(t, s)(-M^2 x(s) + f(s, x(s), x'(s)))ds, t \in [0, +\infty).$$

It follows from Lemma 2.2 that $x(t) \in X$ is a fixed point of T if and only if it is a solution of (1),(2) under the assumption

$$e^{-Mt}(-M^2 x(t) + f(t, x(t), x'(t))) \in L^1[0, +\infty).$$

The main tool of this paper is the Leray-Schauder Continuation Principle as follows:

THEOREM 2.1. Let X be a Banach space and $T : X \to X$ be a completely continuous map. If $\{x \mid x \in X, x = \lambda Tx, 0 < \lambda < 1\}$ is bounded, then, T has a fixed pointed on $B \subset X$, where $B = \{x \mid x \in X, \|x\| \leq R\}$ and $R = \sup\{\|x\| \mid x = \lambda Tx, 0 < \lambda < 1\}$.

3 Main Results

We begin with the following result.

LEMMA 3.1. Under the condition (A_1), the operator $T : X \to X$ is completely continuous.

PROOF. First, we show that T is continuous. Let $x \in X$ and $x_n \to x$. We can assume there exists a $r > 0$ such that $\|x\| \leq r$, $\|x_n\| \leq r$. Since $f(t, u, v)$ is continuous on u and v for a.e. $t \in [0, +\infty)$, then, $-M^2 x_n + f(t, x_n, x_n') \to -M^2 x + f(t, x, x')$.

At the same time, $-M^2x_n + f(t, x_n, x_n')$ is $L^1[0, +\infty)$-Carathé odory, then, there exists $\varphi_r(t) \in L^1[0, +\infty)$ such that $\left|-M^2x_n + f(t, x_n, x_n')\right| \le \varphi_r(t)$. By the Lebesgue Dominated Convergence Theorem, we obtain that the mapping T is continuous.

Let $\Omega \subset X$ be bounded, that is, there exists an $r_1 > 0$ such that $\forall x \in \Omega$, we have $\|x\| \le r_1$. Next, we will show that $T\Omega$ is uniformly bounded in X.

Since $-M^2x + f(t, x, x')$ is L^1-Carathéodory, there exists $\varphi_{r_1}(t) \in L^1[0, \infty)$ such that $\left|-M^2x(t) + f(t, x(t), x'(t))\right| \le \varphi_{r_1}(t)$. Then,

$$
\begin{aligned}
|Tx(t)| &\le \int_0^\infty |G(t,s)| \left|(-M^2x(s) + f(s, x(x), x'(s))\right| ds \\
&\le \frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})} \int_0^\infty \left|-M^2x(s) + f(s, x(x), x'(s))\right| ds \\
&\le \frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})} \int_0^\infty \varphi_{r_1}(t) dt =: a, \qquad (11)
\end{aligned}
$$

that is, $\|Tx\|_\infty \le a$. Furthermore,

$$
\begin{aligned}
\|(Tx)'\|_\infty &\le \frac{3 + e^{M\eta} - e^{-M\eta}}{2(1 - e^{-M\eta})} \int_0^\infty \left|-M^2x + f(s, x(s), x'(s))\right| ds \\
&\le \frac{3 + e^{M\eta} - e^{-M\eta}}{2(1 - e^{-M\eta})} \|\varphi_{r_1}(t)\|_1 = Ma. \qquad (12)
\end{aligned}
$$

Then, we obtain $T\Omega$ is uniformly bounded in X.

Next, we will show that the functions from $T\Omega$ are equicontinuous on any compact interval of $[0, \infty)$. By computation, we obtain

$$
|(Tx)''(t)| \le M^2 \frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})} \int_0^\infty \left|-M^2x(s) + f(s, x(s), x'(s))\right| ds = M^2a. \qquad (13)
$$

Let $\varepsilon > 0$, there exists $\delta = \min\{\frac{\varepsilon}{Ma}, \ \frac{\varepsilon}{M^2a}\} > 0$, while $t_1, t_2 \in [0, \infty)$ and $|t_1 - t_2| < \delta$, $\forall x \in \Omega$, we have

$$
|Tx(t_1) - Tx(t_2)| = \left|\int_{t_1}^{t_2} (Tx)'(s) ds\right| \le Ma\,|t_2 - t_1| < \varepsilon,
$$

and

$$
|(Tx)'(t_1) - (Tx)'(t_2)| = \left|\int_{t_1}^{t_2} (Tx)''(s) ds\right| \le M^2a\,|t_2 - t_1| < \varepsilon.
$$

Then, the functions from $T\Omega$ are equicontinuous on any compact interval of $[0, \infty)$.

Since $(Tx)'(t)$ and $(Tx)''(t)$ is bounded on $[0.\infty)$, then,

$$
|Tx(t) - Tx(+\infty)| = \left|\int_t^{+\infty} (Tx)'(s) ds\right| \to 0, \quad t \to +\infty,
$$

$$
|(Tx)'(t) - (Tx)'(+\infty)| = \left|\int_t^{+\infty} (Tx)''(s) ds\right| \to 0, \quad t \to +\infty.
$$

Hence, from the discussion above, $T\Omega$ is relatively compact. Then, we can obtain that the operator $T : X \to X$ is completely continuous.

THEOREM 3.1. Assume (A_1) and the following conditions are satisfied:

(A_2) there exist functions $a(t), b(t), c(t) : [0, +\infty) \to [0, +\infty), a(t), b(t), c(t) \in L^1[0, +\infty)$ such that

$$\left|-M^2 u + f(t, u, v)\right| \le a(t)\,|u| + b(t)\,|v| + c(t), \ a.e.\ t \in [0, +\infty).$$

(A_3) $(3 + e^{M\eta} - e^{-M\eta})(\|a\|_1 + M\,\|b\|_1) < 2M(1 - e^{-M\eta})$

Then the boundary value problem (1),(2) has at least one solution for every $c(t) \in L^1[0, +\infty)$.

PROOF. We consider

$$x(t) = \lambda T x(t), \quad \lambda \in (0, 1), \quad a.e.\ t \in [0, +\infty), \quad x \in X.$$

Then,

$$\begin{aligned}
|x(t)| &= |\lambda T x(t)| \le \int_0^{+\infty} |G(t,s)| \left|-M^2 x(s) + f(s, x(s), x'(s))\right| ds \\
&\le \frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})} \cdot \int_0^{+\infty} \left|-M^2 x(s) + f(s, x(s), x'(s))\right| ds \\
&\le \frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})} \cdot \int_0^{+\infty} [a(t)\,|x(t)| + b(t)\,|x'(t)| + c(t)] dt \\
&\le \frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})} (\|a\|_1 \|x\|_\infty + \|b\|_1 \|x'\|_\infty + \|c\|_1). \qquad (14)
\end{aligned}$$

So, we have

$$\|x\|_\infty \le \frac{\frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})}\,\|b\|_1}{1 - \frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})}\,\|a\|_1} \parallel x' \parallel_\infty + \frac{\frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})}\,\|c\|_1}{1 - \frac{3 + e^{M\eta} - e^{-M\eta}}{2M(1 - e^{-M\eta})}\,\|a\|_1} =: m\,\|x'\|_\infty + n.$$

At the same time,

$$\begin{aligned}
|x'(t)| &= |\lambda (Tx)'(t)| \le \frac{3 + e^{M\eta} - e^{-M\eta}}{2(1 - e^{-M\eta})} (\|a\|_1 \|x\|_\infty + \|b\|_1 \|x'\|_\infty + \|c\|_1) \\
&\le \frac{3 + e^{M\eta} - e^{-M\eta}}{2(1 - e^{-M\eta})} \Big[\frac{(3 + e^{M\eta} - e^{-M\eta})\,\|a\|_1 \|b\|_1}{2M(1 - e^{-M\eta}) - (3 + e^{M\eta} - e^{-M\eta})\,\|a\|_1} \|x'\|_\infty \\
&\quad + \|b\|_1 \|x'\|_\infty + \frac{(3 + e^{M\eta} - e^{-M\eta})\,\|a\|_1 \|c\|_1}{2M(1 - e^{-M\eta}) - (3 + e^{M\eta} - e^{-M\eta})\,\|a\|_1} + \|c\|_1 \Big]. \qquad (15)
\end{aligned}$$

Hence,

$$\|x'\|_\infty \le \frac{M(3 + e^{M\eta} - e^{-M\eta})\,\|c\|_1}{2M(1 - e^{-M\eta}) - (3 + e^{M\eta} - e^{-M\eta})(\|a\|_1 + M\,\|b\|_1)} =: R_1,$$

and

$$\|x\|_\infty \le mR_1 + n.$$

Let $R = \max\{R_1 + 1,\ mR_1 + n + 1\}$. Then $\forall x \in B = \{x \mid\ x = \lambda Tx,\ \lambda \in (0,1)\}$, we have $\|x\| \le R$. At the same time, $\forall x \in B$, $\int_0^\infty e^{-Ms}(-M^2x(s)+f(s,x(s),x'(s)))ds \le \|\varphi_R\|_1 < \infty$. Hence, $-M^2x(t)+f(t,x(t),x'(t)) \in L^1[0,\infty)$. From Lemma 3.1, we have T is a completely continuous map. Then by the Leray-Schauder Continuation Principle, the boundary value problem (1),(2) has at least one solution.

4 Example

Consider the BVP

$$\begin{cases} x''(t) - 25x(t) = \frac{2e^5-2}{(3e^5+e^{10})(2+\sin t)}e^{-t}x' + e^{-3t} \\ x(0) = x(1), \quad \lim\limits_{t\to+\infty} x(t) = 0. \end{cases} \tag{16}$$

Here $M = 5$, $f(t,x,x') = 25x(t) + \frac{2e^5-2}{(3e^5+e^{10})(2+\sin t)}e^{-t}x' + e^{-3t}$. It is obvious that $\left|-M^2x(t) + f(t,x(t),x'(t))\right| \le \frac{2e^5-2}{3e^5+e^{10}}e^{-t}\,|x'| + e^{-3t}$. $\frac{2e^5-2}{3e^5+e^{10}}e^{-t}r + e^{-3t}$ is $L^1[0,+\infty)$ for $r \in [0,\infty)$. By computation, we can obtain (A_3). Then, (16) has a solution.

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