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Bifurcation Analysis Of An Epidemic Model With Delay As The Control Variable*

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Abstract

A SIS epidemic model proposed by Cooke et al. [2] is investigated. Using time delay as the control parameter, we investigate the stability and Hopf bifurcation of the model by analyzing the distribution of the roots of its associated characteristic equation. Then an explicit formula for determining the stability and the direction of bifurcating periodic solutions is derived by normal form theory and center manifold argument. Finally, some numerical simulations are carried out as supporting evidences of our analytic results.

1 Introduction

In the absence of disease, Cooke et al. [2] proposed the following single species population growth model:

$$\dot{N}(t) = B(N(t-T))N(t-T)e^{-d_1 T} - dN(t). \tag{1}$$

Here, $N(t)$ denotes the mature population of the species, and $d_1, d \geq 0$ are the death rates of immature and mature population, respectively. Time delay $T > 0$ is the maturation time. The birth rate function $B(N)$ for $N \in (0,\infty)$ satisfies the following basic assumptions:

(A1) $B(N) > 0$;
(A2) $B(N)$ is continuously differentiable with $B'(N) < 0$;
(A3) $B(0^+) > de^{d_1 T} > B(\infty)$.

Using (1) as the basis population model, Cooke et al. [2] assumed further that disease has entered the population and then constructed and studied the following SIS epidemic model:

$$\begin{aligned} \dot{I}(t) &= \mu(N-I)\frac{I}{N} - (d+\varepsilon+\gamma)I, \\ \dot{N}(t) &= B(N(t-T))N(t-T)e^{-d_1 T} - dN - \varepsilon I. \end{aligned} \tag{2}$$

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Here the total population is divided into susceptible and infective classes, with the size of each class given by $S(t)$ and $I(t)$ respectively. Therefore, $N(t) = S(t) + I(t)$. The parameter $\mu > 0$ is the contact rate constant, $\gamma \geq 0$ is the recovery rate constant and $\varepsilon \geq 0$ is the disease induced death rate constant. When $B(N)N$ is an increasing function, Cooke et al.[2] determined the dynamics of (2) by applying monotone dynamics system theory and the theory for asymptotically autonomous system. However, when $B(N)N$ does not possess the monotonicity property, e.g., when $B(N)$ is the following Rick function

$$B(N) = be^{-aN}, \tag{3}$$

the dynamical behavior of system (2) becomes a difficult problem. The authors in [2] only considered the special case $\varepsilon = 0$ (when the two equations in (2) are decoupled from each other) and established the stability of the endemic equilibrium. Later, using a perturbation technique, their results were extended in Zhao and Zou [13] to the case when ε is sufficiently small. Overall, the dynamics of this SIS model (2) remains largely undetermined (see also [7, 9, 12, 15]).

We mention that, recently, Wei and Zou [12] also considered (2) under (3). They focused on the existence and stability of Hopf bifurcation and their results are based on the choice of b as the bifurcation parameter. Note that it is an important feature of (2) that the delay T also appears in the coefficient (when $d_1 \neq 0$) in addition to its appearance in the unknown function $B(N)$. Therefore, when discussing the stability, we will face a characteristic equation with coefficients dependent on the delay. It is well known that the analysis of such a characteristic equation becomes more complicated and the system will display richer dynamical behavior, when T is chosen as the parameter. In this paper, using T as control variable and assuming $\varepsilon > 0$, we analyze (2) with (3) to gain more knowledge about its dynamics.

The rest of this paper is organized as follows. At first, we analyze the equilibria and stability of (2) through the study of its characteristic equation, which takes the form of an exponential polynomial with delay-dependent coefficients. Using the approach of Beretta and Kuang [1], we will show that under appropriate conditions, the unique positive equilibrium can be destabilized through a Hopf bifurcation and stability switches of stability-instability-stability occur. Then we investigate the stability and direction of bifurcating periodic solutions of (3) by using the normal form theory and center manifold theorem due to Hassard et al. [4]. Finally, some numerical simulations will be given to illustrate our results.

2 Stability and Hopf Bifurcation

In this section we consider the SIS model (2) with the birth function given by (3), that is

$$\begin{cases} \dot{I}(t) = \mu(N - I)\dfrac{I}{N} - (d + \varepsilon + \gamma)I, \\ \dot{N}(t) = be^{-d_1 T}e^{-aN(t-T)}N(t-T) - dN - \varepsilon I. \end{cases} \tag{4}$$

Here, we assume $B(N)$ for $N \in (0, \infty)$ satisfies $(A1)$, $(A2)$ and

(A3′) $B(0^+) > (d + \varepsilon)e^{d_1 T} > de^{d_1 T} > B(\infty)$ with $\varepsilon > 0$.

In [2], the basic reproduction number for (4) has been identified as

$$R_0 = \frac{\mu}{d + \varepsilon + \gamma}.$$

It has been shown that when $R_0 \leq 1$, (4) has the only disease free equilibrium (DFE), which is globally asymptotically stable. When $R_0 > 1$, the DFE becomes unstable and a nontrivial equilibrium (I^*, N^*) is bifurcated from DFE, where

$$N^* = \frac{1}{a} \ln \frac{b}{[d + \varepsilon(1 - 1/R_0)]e^{d_1 T}} \quad \text{and} \quad I^* = (1 - \frac{1}{R_0})N^*,$$

whose stability remains largely unsolved in [2]. For the sake of convenience, let $Q_0 = \frac{1}{R_0}$. Then $R_0 > 1$ is equivalent to $0 < Q_0 < 1$. Now, we investigate the stability and Hopf bifurcation of (4) around (I^*, N^*).

Set

$$T_m = \frac{1}{d_1} \ln \frac{b}{d + \varepsilon}.$$

It is clear that $T \in [0, T_m)$ under the condition $(A3')$. This restriction on T ensures the positivity of nontrivial (I^*, N^*) which will be denoted by EE (endemic equilibrium). The linearization of (4) at EE is

$$\begin{cases} \dot{I}(t) = -\mu(1 - Q_0)I(t) + \mu(1 - Q_0)^2 N(t), \\ \dot{N}(t) = -\varepsilon I(t) - dN(t) + be^{-aN^*} e^{-d_1 T}(1 - aN^*)N(t - T). \end{cases} \tag{5}$$

Thus the characteristic equation associated with (5) is

$$\lambda^2 + a_1\lambda + a_3a_4 + (a_2a_3a_4 + a_2a_3)e^{-\lambda T} = 0, \tag{6}$$

where

$$\begin{aligned} &a_1 = d + \mu(1 - Q_0) > 0, \quad a_2 = a_2(T) = -\left(1 - \ln \tfrac{b}{[d+\varepsilon(1-Q_0)]e^{d_1 T}}\right), \\ &a_3 = d + \varepsilon(1 - Q_0) > 0, \quad a_4 = \mu(1 - Q_0) > 0. \end{aligned}$$

Equation (6) takes the general form

$$P(\lambda) + Q(\lambda|T)e^{-\lambda T} = 0, \tag{7}$$

with

$$P(\lambda) := \lambda^2 + a_1\lambda + a_3a_4 \quad \text{and} \quad Q(\lambda|T) := a_2(T)a_3a_4 + a_2(T)a_3\lambda. \tag{8}$$

When $T = 0$, equation (7) becomes

$$\lambda^2 + [a_1 + a_2(0)a_3]\lambda + [a_3a_4 + a_2(0)a_3a_4] = 0.$$

Note that for all $T \in [0, T_m)$, $\ln \frac{b}{[d+\varepsilon(1-Q_0)]e^{d_1 T}} > 0$ implies $a_2(T) > -1$. Therefore, $a_1 + a_2(T)a_3 > 0$. Meanwhile,

$$a_3a_4 + a_2(T)a_3a_4 = a_3a_4(1 + a_2(T)) > 0.$$

Therefore, all roots of (6) have negative real parts and we have the following result.

LEMMA 1. Assume $(A1), (A2)$ and $(A3')$ hold. Then the equilibrium (I^*, N^*) of (4) is asymptotically stable when $T = 0$.

Now we assume $T > 0$ and regard it as a parameter to obtain finer results on the stability of EE. Note that equation (7) takes the form of a first-degree exponential polynomial equation in λ with the coefficient Q depending on T. Thus, we use the method introduced by Beretta and Kuang [1], which gives the existence of purely imaginary roots of a characteristic equation with delay dependant coefficients (see also [10]).

In order to apply the criterion in [1], we need to verify the following properties for all $T \in [0, T_m)$ and $\omega \in \mathbb{R}^+$:

(i) $P(0) + Q(0|T) \neq 0$;

(ii) $P(\mathrm{i}\omega) + Q(\mathrm{i}\omega|T) \neq 0$;

(iii) $\limsup_{|\lambda|\to\infty, \mathrm{Re}\lambda\geq 0} \left|\frac{Q(\lambda|T)}{P(\lambda)}\right| < 1$;

(iv) $F(\omega|T) := |P(\mathrm{i}\omega)|^2 - |Q(\mathrm{i}\omega|T)|^2$ has a finite number of zeros;

(v) each positive root $\omega(T)$ of $F(\omega|T) = 0$ is continuous and differentiable in T whenever it exists.

In fact, for $T \in [0, T_m)$ and $\omega \in \mathbb{R}^+$, we have

$$P(0) + Q(0|T) = a_3a_4 + a_2(T)a_3a_4 > 0,$$

$$P(\mathrm{i}\omega) + Q(\mathrm{i}\omega|T) = (-\omega^2 + a_3a_4 + a_2a_3a_4) + \mathrm{i}\omega(a_1 + a_2a_3) \neq 0$$

and

$$\lim_{|\lambda|\to\infty} \left|\frac{Q(\lambda|T)}{P(\lambda)}\right| = \lim_{|\lambda|\to\infty} \left|\frac{a_2a_3\lambda + a_2a_3a_4}{\lambda^2 + a_1\lambda + a_3a_4}\right| = 0.$$

Therefore, (i), (ii) and (iii) are fulfilled.

Let $F(\omega|T)$ be defined as in (iv). From (7) and the definitions of $P(\lambda)$ and $Q(\lambda|T)$, we have

$$F(\omega|T) = \omega^4 + (a_1^2 - 2a_3a_4 - a_2^2(T)a_3^2)\omega^2 + a_3^2a_4^2(1 - a_2^2(T)).$$

It is obvious that property (iv) is satisfied. Finally, (v) is satisfied since $F(\omega|T)$ is a quadratic polynomial in ω^2 and the fact that $a_i(\tau), i = 1, 2, 3, 4$ are all continuous functions of τ.

Now let $\lambda = \mathrm{i}\omega(\omega > 0)$ be a root of equation (6). Substituting it into (6) and separating the real and imaginary parts, we obtain

$$\begin{aligned} \omega^2 - a_3a_4 &= a_2a_3\omega\sin\omega T + a_2a_3a_4\cos\omega T, \\ a_1\omega &= a_2a_3a_4\sin\omega T - a_2a_3\omega\cos\omega T. \end{aligned}$$

It follows that

$$\begin{aligned} \sin\omega T &= \frac{\omega(\omega^2 - a_3a_4 + a_1a_4)}{a_2a_3(\omega^2 + a_4^2)}, \\ \cos\omega T &= -\frac{\omega^2(a_1 - a_4) + a_3a_4^2}{a_2a_3(\omega^2 + a_4^2)}, \end{aligned} \tag{9}$$

which yields

$$F(\omega|T) = 0.$$

On the other hand, the polynomial function F can be written as

$$F(\omega|T) = h(\omega^2|T),$$

where h is a second-degree polynomial defined by

$$h(Z|T) = Z^2 + b_1(T)Z + b_2(T).$$

Here,

$$\begin{array}{rcl} b_1(T) & = & a_1^2 - 2a_3a_4 - a_2^2(T)a_3^2, \\ b_2(T) & = & a_3^2a_4^2 - a_2^2(T)a_3^2a_4^2. \end{array}$$

Moreover, set

$$\Delta(T) = b_1^2(T) - 4b_2(T).$$

Then, when $\Delta(T) \geq 0$, $h(Z|T) = 0$ has a pair of real roots given by

$$Z_+(T) = \frac{-b_1(T) + \sqrt{\Delta(T)}}{2},$$
$$Z_-(T) = \frac{-b_1(T) - \sqrt{\Delta(T)}}{2}.$$

Assume

(A4) $(1 - Q_0)(\mu^2 - 2\varepsilon\mu - a_2^2(T)\varepsilon^2) - 2d\varepsilon a_2^2(T) \geq 0.$

Then $b_1(T) \geq 0$. Therefore, $h(Z|T) = 0$ has a positive root if and only if $b_2(T) < 0$. Note that

$$b_2(T) = a_3^2a_4^2(1 + a_2(T))(1 - a_2(T))$$

and

$$1 - a_2(T) = 2 - \ln \frac{b}{[d + \varepsilon(1 - Q_0)]e^{d_1T}}.$$

These imply $b_2(T) < 0$ if, and only if, the following condition holds:

(A5) $2 - \ln \frac{b}{[d+\varepsilon(1-Q_0)]e^{d_1T}} < 0,$

where $1 + a_2(T) > 0$ is used. Therefore, we have

LEMMA 2. If $(A4)$ and $(A5)$ are satisfied, then $h(Z|T) = 0$ has only one positive root denoted by Z_+. Accordingly, $F(\omega|T) = 0$ has a unique positive root given by $\omega = \sqrt{Z_+}$.

Assume further that

(A6) $b > [d + \varepsilon(1 - Q_0)]e^2$

to make sure that (A5) is nonvacuous. Under (A6), denote

$$T^0 = \max\big\{\{\tau \geq 0 : \text{(A4) and (A5) hold}\} \cap [0, T_m)\big\}.$$

Therefore, there exists an $\omega = \omega(T) > 0$ such that $F(\omega|T) = 0$ for $T \in (0, T^0)$. Let $\theta(T) \in [0, 2\pi]$ be defined for $T \in (0, T^0)$ by

$$\begin{aligned} \sin\theta(T) &= \frac{\omega(\omega^2 - a_3a_4 + a_1a_4)}{a_2(T)a_3\omega^2 + a_2(T)a_3a_4^2}, \\ \cos\theta(T) &= -\frac{\omega^2(a_1 - a_4) + a_3a_4^2}{a_2(T)a_3\omega^2 + a_2(T)a_3a_4^2}, \end{aligned} \tag{10}$$

where $\omega = \omega(T)$ for $T \in (0, T^0)$. $\theta(T)$ is well and uniquely defined for all $T \in (0, T^0)$ (see [1]). Noticing that $a_1 - a_4 = d > 0$ and

$$-a_3a_4 + a_1a_4 = a_4(1 - Q_0)(\mu - \varepsilon) > 0,$$

we have $\theta(T) \in (\frac{\pi}{2}, \pi)$.

From the definition of $\theta(T)$, it is known that $\omega(T)T = \theta(T) + 2n\pi$. We may check that $\mathrm{i}\omega^*(\omega^* > 0)$ is a purely imaginary root of (6) if, and only if, T^* is a zero of the function S_n defined by

$$S_n(T) = T - \frac{\theta(T) + 2n\pi}{\omega(T)}, \quad T \in (0, T_0), \quad n \in \mathbb{N}. \tag{11}$$

The following theorem is due to Beretta and Kuang [1].

THEOREM 1. Assume that $S_n(T)$ has a positive zero $T^* \in (0, T_0)$ for some $n \in \mathbb{N}$. Then a pair of simple purely imaginary roots $\pm \mathrm{i}\omega(T^*)$ of (6) exists at $T = T^*$, and

$$\text{Sign}\left\{\frac{\mathrm{d}\,\mathrm{Re}\,(\lambda)}{\mathrm{d}T}\Big|_{\lambda=\mathrm{i}\omega(T^*)}\right\} = \text{Sign}\left\{\frac{\partial F}{\partial \omega}(\omega(T^*))\right\} \times \text{Sign}\left\{\frac{\mathrm{d}\,K_n(T)}{\mathrm{d}T}\Big|_{T=T^*}\right\}. \tag{12}$$

Since $\frac{\partial F}{\partial \omega}(\omega^*) = 2\omega^*\sqrt{\Delta(T^*)} > 0$, condition (12) is equivalent to

$$\delta(T^*) = \text{Sign}\left\{\frac{\mathrm{dRe}(\lambda)}{\mathrm{d}T}\Big|_{\lambda=\mathrm{i}\omega(T^*)}\right\} = \text{Sign}\left\{\frac{\mathrm{d}S_n(T)}{\mathrm{d}T}\Big|_{T=T^*}\right\}.$$

Therefore, this pair of simple conjugate purely imaginary roots crosses the imaginary axis from left to right if $\delta(T^*) > 0$, and crosses the imaginary axis from right to left if $\delta(T^*) < 0$.

By the definition of $\omega(T)$, we know $\lim_{T\to 0^+}\omega(T)$ is a positive number. Since $\theta(T) \in (\frac{\pi}{2}, \pi)$, when $T \to 0^+$, $S_n(T) = T - \frac{\theta(T)+2n\pi}{\omega(T)} < 0$. Moreover, for all $T \in (0, T^0)$, $S_n(T) > S_{n+1}(T)$ with $n \in \mathbb{N}$. Therefore, if $S_0(T)$ has no zeros in $(0, T^0)$, then $S_n(T)$ have no zeros in $(0, T^0)$ for all $n \in \mathbb{N}$, and if the function $S_n(T)$ has a positive zero $T \in (0, T^0)$ for some $n^* \in \mathbb{N}$, there exists at least one zero satisfying

$$S_n(T^*) = 0 \ \text{ and } \ \frac{\mathrm{d}\,S_n(T^*)}{\mathrm{d}T} > 0 \text{ with } n \le n^*.$$

In addition, when $T \to T^0$, $\omega(T) \to 0$ and $\theta(T) \to \pi$ by the facts that $\sin\theta(T) \to 0$ and $\cos\theta(T) \to -1$. Therefore, we have by (11) that

$$\lim_{T\to T^0} S_n(T) = -\infty.$$

Let

$$\Gamma = \{T \in [0, T^0) | S_n(T) = 0,\ n \in \mathbb{N}\}$$

and let its minimum and maximum be $T_{\min}$ and $T_{\max}$ respectively. Now, we may conclude the following existence of a Hopf bifurcation.

THEOREM 2. Assume $(A1)$, $(A2)$ and $(A3')$ hold.

(i) If $(A6)$ does not hold, then the steady-state (I^*, N^*) of (4) is asymptotically stable for all $T \in [0, T_m)$.

(ii) Assume in addition that $(A4)$-$(A6)$ hold. Then

(iia) if $S_0(T)$ has no positive zeros in $(0, T^0)$, then (I^*, N^*) is asymptotically stable for all $T \in [0, T_m)$;

(iib) if $\Gamma \neq \emptyset$ and $S_n'(T) \neq 0$ for $T \in \Gamma$ and $n \in \mathbb{N}$, then (I^*, N^*) is asymptotically stable for $T \in [0, T_{\min}) \cup (T_{\max}, T^0)$ and unstable for $T \in (T_{\min}, T_{\max})$ with a Hopf bifurcation occurring when $T \in \Gamma$.

REMARK 1. Theorem 2(iib) gives the sufficient condition for (2) to have phenomenon of stability switches of stability-instability-stability.

3 The Properties of Hopf Bifurcation

Theorem 2(iib) gives sufficient conditions to ensure (4) undergoes a Hopf bifurcation at (I^*, N^*). In this section, under the conditions of Theorem 2(iib), we shall use the center manifold and normal form theory presented by Hassard et al. [4] to study the direction of Hopf bifurcation and the stability of the bifurcating periodic solutions from (I^*, N^*) (see also [5, 6, 8, 11, 14]). Without loss of generality, let T^* be any critical value of Hopf bifurcation. Setting $T = T^* + \nu$, then $\nu = 0$ is a Hopf bifurcation critical value of (4).

Let $x = I - I^*$ and $y = N - N^*$. Then we rescale the time by $t \to (t/T)$ to normalize the delay so that system (4) can be written in the form:

$$\begin{aligned}
\dot{x}(t) \quad = \quad & \Big[-\mu(1-Q_0)x(t) + \mu(1-Q_0)^2 y(t) - \tfrac{\mu}{N^*}x^2(t) - \tfrac{\mu(1-Q_0)^2}{N^*}y^2(t) \\
& + \tfrac{2\mu(1-Q_0)}{N^*}x(t)y(t)] + \tfrac{\mu(1-Q_0)^2}{{N^*}^2}y^3(t) + \tfrac{\mu}{{N^*}^2}x^2(t)y(t) \\
& - \tfrac{2\mu(1-Q_0)}{{N^*}^2}x(t)y^2(t) + \cdots \Big](T^* + \nu), \\
\dot{y}(t) \quad = \quad & \Big[-\varepsilon x(t) - dy(t) + be^{-(d_1 T + aN^*)}((1-aN^*)y(t-1) - \tfrac{a(2-aN^*)}{2}y^2(t-1) \\
& + \tfrac{a^2(3-aN^*)}{6}y^3(t-1) + \cdots \Big](T^* + \nu).
\end{aligned} \tag{13}$$

For $\varphi = col(\varphi_1, \varphi_2) \in C := C([-1, 0], \mathrm{R}^2)$, define

$$\begin{aligned}
L_\nu(\varphi) = & (T^* + \nu) \begin{bmatrix} -\mu(1-Q_0) & \mu(1-Q_0)^2 \\ -\varepsilon & -d \end{bmatrix} \begin{bmatrix} \varphi_1(0) \\ \varphi_2(0) \end{bmatrix} \\
& + (T^* + \nu) \begin{bmatrix} 0 & 0 \\ 0 & b(1-aN^*)e^{-(d_1 T^* + d_1 \nu + aN^*)} \end{bmatrix} \begin{bmatrix} \varphi_1(-1) \\ \varphi_2(-1) \end{bmatrix} \\
& \stackrel{\text{def}}{=} (T^* + \nu)A\phi(0) + (T^* + \nu)B\phi(-1).
\end{aligned}$$

By the Riesz representation theorem, there exists a 2×2 matrix, $\eta(\theta,\nu)(\theta\in[-1,0])$, whose elements are functions of bounded variation such that

$$L_\nu\varphi=\int_{-1}^{0}\mathrm{d}\eta(\theta,\mu)\varphi(\theta), \tag{14}$$

with any $\varphi\in C$. In fact, we can choose

$$\eta(\theta,\mu)=\begin{cases}(T^*+\nu)A, & \theta=0,\\ 0, & \theta\in(-1,0),\\ -(T^*+\nu)B, & \theta=-1.\end{cases}$$

Define operators

$$A(\nu)\varphi=\begin{cases}\mathrm{d}\varphi(\theta)/\mathrm{d}\theta, & \theta\in[-1,0),\\ \int_{-1}^{0}\mathrm{d}\eta(t,\nu)\varphi(t), & \theta=0,\end{cases} \tag{15}$$

and

$$R(\nu)\varphi=\begin{cases}0, & \theta\in[-1,0),\\ F(\nu,\varphi), & \theta=0,\end{cases}$$

where

$$F(\nu,\varphi)=(T^*+\nu)\begin{bmatrix}-\frac{\mu}{N^*}\varphi_1^2(0)-\frac{\mu(1-Q_0)^2}{N^*}\varphi_2^2(0)+\frac{2\mu(1-Q_0)}{N^*}\varphi_1(0)\varphi_2(0)+\frac{\mu(1-Q_0)^2}{N^{*2}}\varphi_2^3(0)\\ +\frac{\mu}{N^*}\varphi_1^2(0)\varphi_2(0)-\frac{2\mu(1-Q_0)}{N^{*2}}\varphi_1(0)\varphi_2^2(0)+\cdots\\ \\ be^{-(d_1T^*+d_1\nu+aN^*)}\big(-\frac{a(2-aN^*)}{2}\varphi_2^2(-1)+\frac{a^2(3-aN^*)}{6}\varphi_2^3(-1)+\cdots\big)\end{bmatrix}.$$

Then system (4) is equivalent to the following operator equation:

$$\dot{u}_t=A(\nu)u_t+R(\nu)u_t, \tag{16}$$

where $u=col(x,y)$ and $u_t=u(t+\theta)$ for $\theta\in[-1,0]$.

For $\psi\in C^1([0,1],(C^2)^*)$, define

$$A^*\psi(s)=\begin{cases}-\mathrm{d}\psi(s)/\mathrm{d}s, & s\in(0,1],\\ \int_{-1}^{0}\mathrm{d}\eta(t,0)\psi(-t), & s=0,\end{cases}$$

and a bilinear form

$$\langle\psi,\varphi\rangle=\bar{\psi}(0)\varphi(0)-\int_{-1}^{0}\int_{0}^{\theta}\bar{\psi}(\xi-\theta)\mathrm{d}\eta(\theta)\varphi(\xi)\mathrm{d}\xi$$

for any $\phi\in C([-1,0],\mathbb{R}^2)$ and $\psi\in C([0,1],(\mathbb{R}^2)^*)$, where $\eta(\theta)=\eta(\theta,0)$. Then A^* and $A(0)$ are adjoint operators.

It is not difficult to verify that vectors $q(\theta)=col(q_1,q_2)$ ($\theta\in[-1,0]$) and $q^*(s)=\frac{1}{D}(q_1^*,q_2^*)$ ($s\in[0,1]$) are the eigenvectors of A^* and $A(0)$ corresponding to the eigenvalues $\mathrm{i}\omega^*T^*$ and $-\mathrm{i}\omega^*T^*$, respectively, where

$$\begin{aligned}col(q_1,q_2)&=col\left(1,\frac{T^*\mu(1-Q_0)+\mathrm{i}\omega^*}{T^*\mu(1-Q_0)^2}\right)e^{\mathrm{i}\omega^*T^*\theta},\\ (q_1^*,q_2^*)&=\left(1,-\frac{T^*\mu(1-Q_0)-\mathrm{i}\omega^*}{T^*\varepsilon}\right)e^{\mathrm{i}\omega^*T^*s}.\end{aligned}$$

Let

$$D = \bar{q_1^*} q_1 + \bar{q_2^*} q_2 + \bar{q_2^*} q_2 b(1 - aN^*) T^* e^{-(d_1 T^* + aN^* + \mathrm{i}\omega^* T^*)}.$$

Then $\langle q^*(s), q(\theta)\rangle = 1$ and $\langle q^*(s), \bar{q}(\theta)\rangle = 0$.

Set

$$\alpha = \frac{T^*\mu(1-Q_0) + \mathrm{i}\omega^*}{T^*\mu(1-Q_0)^2} \quad \text{and} \quad \beta = \frac{T^*\mu(1-Q_0) - \mathrm{i}\omega^*}{T^*\mu(1-Q_0)^2}.$$

Following the algorithms given in [4] and using a computation process similar to [4], we can obtain the coefficients:

$$\begin{aligned}
g_{20} &= \tfrac{2T^*}{D}\Big[-2\Big(\tfrac{\mu}{N^*} + \tfrac{(T^*\mu(1-Q_0)+\mathrm{i}\omega^*)^2}{T^*N^*\mu(1-Q_0)^2} - 2\tfrac{T^*\mu(1-Q_0)+\mathrm{i}\omega^*}{T^*N^*(1-Q_0)}\Big) + be^{-(d_1T^*+aN^*)}a(2-aN^*) \\
&\quad \times \tfrac{(T^*\mu(1-Q_0)+\mathrm{i}\omega^*)^2}{\mu^2 {T^*}^2(1-Q_0)^4} e^{-2\mathrm{i}\omega^* T^*}\Big], \\
g_{11} &= \tfrac{T^*}{D}\Big[\tfrac{2\mu}{N^*} - \tfrac{2\mu^2{T^*}^2(1-Q_0)^2+{\omega^*}^2)^2}{N^*T^*\mu(1-Q_0)^2} - be^{-(d_1T^*+aN^*)}a(2-aN^*)\tfrac{\mu^2{T^*}^2(1-Q_0)^2+{\omega^*}^2}{\mu^2{T^*}^2(1-Q_0)^4}\Big], \\
g_{02} &= \tfrac{2T^*}{D}\Big[-2\Big(\tfrac{\mu}{N^*} + \tfrac{(T^*\mu(1-Q_0)-\mathrm{i}\omega^*)^2}{T^*N^*\mu(1-Q_0)^2} - 2\tfrac{T^*\mu(1-Q_0)-\mathrm{i}\omega^*}{T^*N^*(1-Q_0)}\Big) + be^{-(d_1T^*+aN^*)} \\
&\quad \times a(2-aN^*)\tfrac{(T^*\mu(1-Q_0)-\mathrm{i}\omega^*)^2}{\mu^2{T^*}^2(1-Q_0)^4} e^{2\mathrm{i}\omega^*T^*}\Big], \\
g_{21} &= \tfrac{-2T^*}{D}\Big[\tfrac{-\mu}{N^*}(w_{20}^{(1)} + 2w_{11}^{(1)} + (1-Q_0)^2(2\alpha w_{11}^{(2)} + \beta w_{20}^{(2)}) - 2(1-Q_0) \\
&\quad \times (w_{11}^{(2)} + \tfrac{w_{20}^{(2)}}{2} + \alpha w_{11}^{(1)} + \tfrac{\beta}{2}w_{20}^{(1)}) - \tfrac{(1-Q_0)^2}{N^*}3\alpha^2\beta - \tfrac{1}{N^*}(2\alpha+\beta) + \tfrac{2(1-Q_0)}{N^*}(\alpha^2+\beta)) \\
&\quad \times be^{-(d_1T^*+aN^*)}\tfrac{T^*\mu(1-Q_0)+\mathrm{i}\omega^*}{2T^*\varepsilon}(a(aN^*-2)(2\alpha w_{11}^{(2)}(-1)e^{-\mathrm{i}\omega^*T^*} \\
&\quad + \beta w_{20}^{(2)}(-1)e^{\mathrm{i}\omega^*T^*}) + a^2(3-aN^*)\alpha^2\beta e^{-\mathrm{i}\omega^*T^*})\Big],
\end{aligned}$$

where

$$\begin{aligned}
w_{20}(\theta) &= \frac{-ig_{20}}{\omega^*T^*} q(0) e^{\mathrm{i}\omega^*T^*\theta} + \frac{i\bar{g}_{20}}{3\omega^*T^*}\bar{q}(0)e^{-\mathrm{i}\omega^*T^*\theta} + E_1 e^{2\mathrm{i}\omega^*T^*\theta}, \\
w_{11}(\theta) &= \frac{-ig_{11}}{\omega^*T^*} q(0) e^{\mathrm{i}\omega^*T^*\theta} + \frac{i\bar{g}_{11}}{\omega^*T^*}\bar{q}(0)e^{-\mathrm{i}\omega^*T^*\theta} + E_2,
\end{aligned}$$

and

$$\begin{aligned}
E_1 &= 2\begin{pmatrix} -\mu(1-Q_0) - 2\mathrm{i}\omega^*T^* & \mu(1-Q_0)^2 \\ -\varepsilon & -d + b(1-aN^*)e^{-(d_1T^*+aN^*)} - 2\mathrm{i}\omega^*T^* \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} \frac{\mu}{N^*} + \frac{(T^*\mu(1-Q_0)+\mathrm{i}\omega^*)^2}{T^*N^*\mu(1-Q_0)^2} - 2\frac{T^*\mu(1-Q_0)+\mathrm{i}\omega^*}{T^*N^*(1-Q_0)} \\ be^{-(d_1T^*+aN^*)}\frac{a(2-aN^*)}{2}\left(\frac{T^*\mu(1-Q_0)+\mathrm{i}\omega^*}{T^*\mu(1-Q_0)^2}\right)^2 e^{-2\mathrm{i}\omega^*T^*} \end{pmatrix}, \\
E_2 &= \begin{pmatrix} -\mu(1-Q_0) & \mu(1-Q_0)^2 \\ -\varepsilon & -d + b(1-aN^*)e^{-(d_1T^*+aN^*)} \end{pmatrix}^{-1} \\
&\quad \times \begin{pmatrix} -\frac{2\mu}{N^*} + \frac{2\mu^2{T^*}^2(1-Q_0)^2+{\omega^*}^2)^2}{N^*\mu(1-Q_0)^2} \\ -be^{-(d_1T^*+aN^*)}a(aN^*-2)\frac{\mu^2{T^*}^2(1-Q_0)^2+{\omega^*}^2}{\mu^2{T^*}^2(1-Q_0)^4} \end{pmatrix}.
\end{aligned}$$

Consequently, we can compute the following quantities:

$$\begin{aligned} C_1(0) &= \frac{\mathrm{i}}{2\omega^* T^*}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{\mathrm{Re}\,\{C_1(0)\}}{\mathrm{Re}\,\lambda'(T^*)}, \\ \beta_2 &= 2\mathrm{Re}\,\{C_1(0)\}, \\ T_2 &= -\frac{\mathrm{Im}\,C_1(0) + \mu_2 \mathrm{Im}\,(\lambda'(T^*))}{\omega^* T^*}, \end{aligned}$$

which determine the properties of bifurcating periodic solutions. To be concrete, μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0$ (resp. < 0), then Hopf bifurcations are forward (resp. backward), that is, the bifurcating periodic solutions exist for $T > T^*$ (resp. $< T^*$). β_2 determines the stability of the bifurcating periodic solution: if $\beta_2 < 0$ (resp. > 0), then the bifurcating periodic solutions are stable (resp. unstable). T_2 determines the periods of the bifurcating periodic solutions: the periods increase (resp. decrease) if $T_2 > 0$ (resp. < 0).

Theorem 1 shows $\mathrm{Sign}\,\mathrm{Re}\,(\lambda'(T^*)) = \mathrm{Sign}\,S_n'(T^*)$. Thus we have the following result.

THEOREM 3. The Hopf bifurcations of the system (4) at the positive equilibrium (I^*, N^*) when $T = T^*$ are forward (resp. backward) if $\mathrm{Sign}\,\{S_n'(T^*) \cdot \mathrm{Re}\,C_1(0)\} = 1$ (resp. -1) and the bifurcating periodic solutions on the center manifold are stable (unstable) if $\mathrm{Re}\,C_1(0) < 0$ (resp. > 0). In particular, if $T^* = T_{\min}$ and $S_0'(T_{min}) \neq 0$, then the periodic solution bifurcated from (I^*, N^*) is stable (resp. unstable) if $\mathrm{Re}\,C_1(0) < 0$ (resp. > 0).

4 Numerical Simulations

In this section, we shall carry out some numerical simulations to support our theoretical analysis. There are eight parameters involved in (4) including the delay T. Next, we choose two sets of parameters, under which $(A1), (A2)$ and $(A3')$ are satisfied:

(a) $\mu = 6, d = 0.1, \varepsilon = 0.1, \gamma = 3.5, b = 12, a = 10$ and $d_1 = 0.1$;

(b) $\mu = 6, d = 0.1, \varepsilon = 0.1, \gamma = 3.5, b = 16, a = 10$ and $d_1 = 0.3$.

Under (a), $(A4)$-$(A6)$ are met. We draw the graph of S_0 and S_1 versus T on $T \in [0, T^0)$ in Figure 1 (left) with $T^0 = 24.63$. In this case, there are only two critical values of T denoted by $T_{\min} \approx 4.8357$ and $T_{\max} \approx 21.808$, respectively. Theorem 2(iib) tells that a scenario and Hopf bifurcation occur, that is

(1) EE is asymptotically stable for $T \in [0, T_{\min}) \bigcup \ (T_{\max}, T^0)$;

(2) EE becomes unstable for $T \in (T_{\min}, T_{\max})$;

(3) when $T = T_{\min}$ or $T_{\max}$, there occurs a Hopf bifurcation at EE.

Figure 2 describes the case in (1) with $T = 2.5$. In addition, when $T \approx 4.8357$ or 21.808, using the formula given in Section 3, we compute $\mathrm{Re}\,C_1(0)\ |_{T=T_{\min}} \approx -271.176 < 0$ and $\mathrm{Re}\,C_1(0)\ |_{T=T_{\max}} \approx -188.5106 < 0$, respectively. Consequently, when $T = T_{min}$ (resp. $T = T_{max}$), the Hopf bifurcation of system (4) at EE is forward (backward) and the bifurcating periodic solution is stable. Figure 3 with $T = 5.195$ and Figure 4 with $T = 21$ describe the existence of stable periodic solution of (4).

Comparatively, under (b), (A4)-(A6) hold. We plot the curve of S_0 on the interval $[0, T^0)$ with $T^0 = 9.1689$ (see Figure 1 (right)). It can be observed that S_n has no zeros for arbitrary $n \in \mathbb{N}$, satisfying the condition in Theorem 2(iia). Therefore, EE is asymptotically stable for $T \in [0, T^0)$.

To sum up, it seems that if b and d increase, then the oscillatory dynamical behavior (under (a)) becomes stable (under (b)).

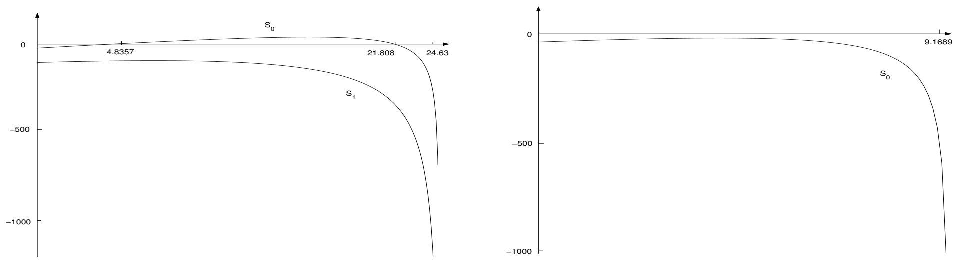

Figure 1: Graph of $S_n(T)$ on $[0, T^0)$. Left: S_0 and S_1 with parameters given in (a); Right: S_0 with parameters given in (b)

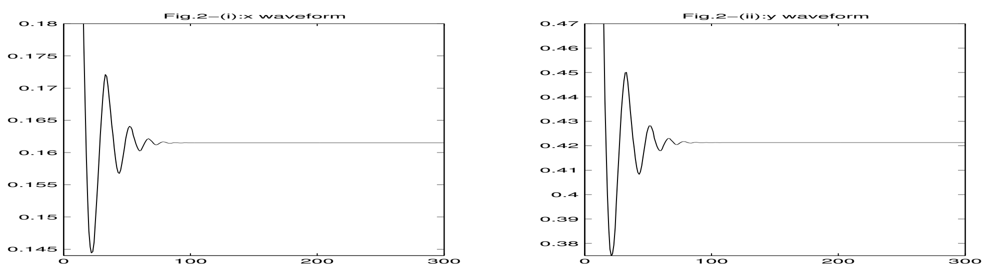

Figure 2: EE is stable under (a) with $T = 2.5$

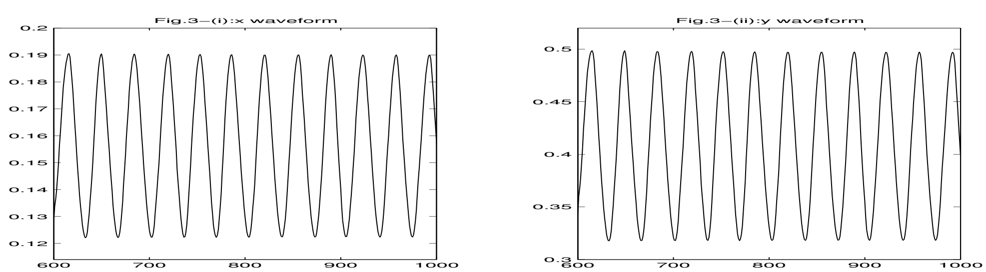

Figure 3: A periodic solution of system (4) under (a) with $T = 5.195$

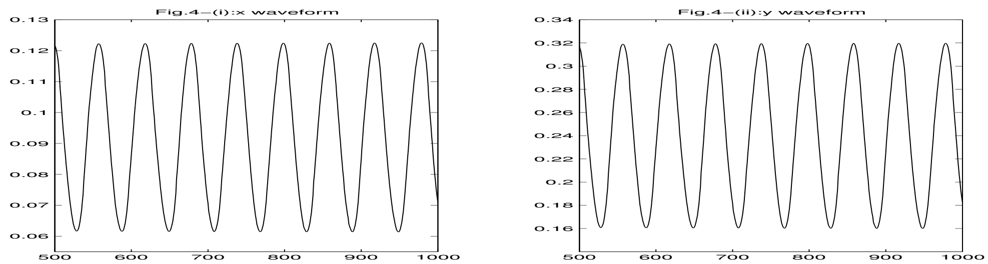

Figure 4: A periodic solution of system (4) under (a) with $T = 21$

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