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Parametric Two-Point Integral Inequalities For n-Time Differentiable Functions With Applications*

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Abstract

New parametric two point integral inequalities for n-time differentiable functions are presented. These inequalities are used to obtain some new estimations for the remainder in Taylor's formula. New inequalities for the expectation and variance of a random variable defined on a finite interval are also given.

1 Introduction

In the literature on numerical integration, the following estimation is well known as the trapezoid inequality:

$$\left|\frac{f(b)+f(a)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)^2}{12}\sup_{x\in(a,b)}|f''(x)|, \tag{1}$$

where the mapping $f:[a,b]\to\mathbf{R}$ is twice differentiable on the interval (a,b), with the second derivative bounded on (a,b). For more results on the trapezoid inequality and their applications we refer to [3], while in [2] we can find the following result: Let $f\in C^n[a,b]$ be a function such that $f^{(n+1)}$ is integrable and bounded on (a,b). Then for any positive number ρ the following estimation holds:

$$\begin{aligned}
&\frac{(1+\rho)(b-a)^{n+1}}{(n+2)!\,(n+1)}\inf_{x\in(a,b)}f^{(n+1)}(x)\\
\leq\;& -\frac{\rho+(-1)^{n+1}}{b-a}\int_a^b f(x)\,dx+\frac{\rho f(b)+(-1)^{n+1}f(a)}{n+1}\\
&+\sum_{k=0}^{n-1}\frac{(n-k)(b-a)^k}{(n+1)(k+1)!}\left(\rho f^{(k)}(a)+(-1)^{n+k+1}f^{(k)}(b)\right)\\
\leq\;& \frac{(1+\rho)(b-a)^{n+1}}{(n+2)!\,(n+1)}\sup_{x\in(a,b)}f^{(n+1)}(x).
\end{aligned} \tag{2}$$

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This inequality is used in [2] in order to obtain several inequalities for n-time differentiable functions, as for example some generalizations of inequality (1).

In what follows, we present some two point integral inequalities for n-time differentiable functions involving two parameters which are generalizations of inequality (2). As applications to that some new interesting integral inequalities are given which are being used to obtain some estimations for the expectation and the variance of a random variable defined on a finite interval. The results presented here are related to the ones obtained in paper [1]. New inequalities involving the remainder in Taylor's formula are also presented which give better approximations compared to the classical Taylor's expansion.

2 Main Results

We begin with the following result.

THEOREM 1. Assume (p,q) is a pair of conjugate exponents, that is $1 \le p,q \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$. Let $g:[a,b]\to \mathbf{R}$ be a $(n+1)$-time differentiable function on (a,b) for some $n \ge 1$ with $g^{(n+1)}$ integrable on (a,b). Then for $n \ge 2$ and all $\alpha,\beta \in \mathbf{R}$ we have the inequalities

$$\begin{aligned}&\Bigg|\frac{\alpha+\beta}{b-a}\int_a^b g(x)\,dx-(\alpha g(a)+\beta g(b))\\&-\sum_{k=1}^{n-1}\frac{n-k}{n^2-1}\left((n\alpha-\beta)\,g^{(k)}(a)+(-1)^k(n\beta-\alpha)\,g^{(k)}(b)\right)\frac{(b-a)^k}{(k+1)!}\Bigg|\\&\le\begin{cases}\frac{(|n\alpha-\beta|+|n\beta-\alpha|)(b-a)^{n+1}}{(n+2)!(n^2-1)}\left\|g^{(n+1)}\right\|_\infty & \text{if } g^{(n+1)}\in L_\infty[a,b]\\ \frac{(|n\alpha-\beta|^q+|n\beta-\alpha|^q)^{\frac{1}{q}}(b-a)^{n+\frac{1}{q}}}{(n-1)(n+1)![(nq+1)(nq+2)]^{\frac{1}{q}}}\left\|g^{(n+1)}\right\|_p & \text{if } g^{(n+1)}\in L_p[a,b],\ 1<p<\infty\\ \frac{(b-a)^{n-1}}{(n-1)(n+1)!}\left\|g^{(n+1)}\right\|_1\max\{|n\alpha-\beta|,|n\beta-\alpha|\} & \text{if } g^{(n+1)}\in L_1[a,b]\end{cases}\end{aligned}\tag{3}$$

and the first inequality in (3) is sharp in the following two cases: i) n is odd and $\frac{\beta}{\alpha}\in\left[\frac{1}{n},n\right]$, ii) n is even and $\alpha=0$ or $\frac{\beta}{\alpha}\notin\left[\frac{1}{n},n\right]$.

PROOF. Let $K_n(u,x)$ be the kernel given by

$$K_n(u,x):=\begin{cases}\frac{\alpha-n\beta}{n-1}(a-u)^n & \text{if} \quad u\in[a,x]\\ \frac{n\alpha-\beta}{n-1}(b-u)^n & \text{if} \quad u\in(x,b]\end{cases},\ n>1.$$

Then, using twice the Taylor's formula with an integral form of remainder we easily get

$$\begin{aligned}
&\frac{1}{(n+1)!\,(b-a)}\int_a^b\int_a^b K_n(u,x)\,g^{(n+1)}(u)\,dudx \\
= &\frac{n\beta-\alpha}{(n^2-1)(b-a)}\int_a^b\int_x^a \frac{(a-u)^n}{n!}g^{(n+1)}(u)\,dudx \\
&+\frac{n\alpha-\beta}{(n^2-1)(b-a)}\int_a^b\int_x^b \frac{(b-u)^n}{n!}g^{(n+1)}(u)\,dudx \\
= &\frac{n\beta-\alpha}{n^2-1}g(a)+\frac{n\alpha-\beta}{n-1}g(b)-\frac{(\alpha+\beta)}{(n+1)(b-a)}\int_a^b g(x)\,dx-\frac{1}{b-a}\sum_{k=1}^n I_k, \quad (4)
\end{aligned}$$

where

$$I_k := \frac{1}{k!}\int_a^b g^{(k)}(x)\left(\frac{n\beta-\alpha}{n^2-1}(a-x)^k+\frac{n\alpha-\beta}{n^2-1}(b-x)^k\right)dx,\ \ k\geq 0.$$

Now, using integration by parts we have,

$$I_{k+1}-I_k=-\left(\frac{n\alpha-\beta}{n^2-1}g^{(k)}(a)+(-1)^k\,\frac{n\beta-\alpha}{n^2-1}g^{(k)}(b)\right)\frac{(b-a)^{k+1}}{(k+1)!},\ \ k\geq 0. \qquad (5)$$

Moreover, it is easy to verify that any sequence (I_n), $n\geq 0$ the following identity holds

$$\sum_{k=1}^n I_k = nI_0+\sum_{k=0}^{n-1}(n-k)(I_{k+1}-I_k). \qquad (6)$$

Combining (4) with (5) and (6) we easily get the following identity:

$$\begin{aligned}
&\frac{1}{(n+1)!\,(b-a)}\int_a^b\int_a^b K_n(u,x)\,g^{(n+1)}(u)\,dudx \\
= &\alpha g(a)+\beta g(b)-\frac{\alpha+\beta}{b-a}\int_a^b g(x)\,dx \\
&+\sum_{k=1}^{n-1}\frac{n-k}{n^2-1}\left((n\alpha-\beta)\,g^{(k)}(a)+(-1)^k(n\beta-\alpha)\,g^{(k)}(b)\right)\frac{(b-a)^k}{(k+1)!}. \qquad (7)
\end{aligned}$$

Now we will use identity (7) to prove Theorem 1.

If $g^{(n+1)}\in L_\infty[a,b]$, then it is not difficult to get:

$$\begin{aligned}
&\frac{1}{(n+1)!\,(b-a)}\left|\int_a^b\int_a^b K_n(u,x)\,g^{(n+1)}(u)\,dudx\right| \\
\leq &\frac{(|\alpha-n\beta|+|n\alpha-\beta|)(b-a)^{n+1}}{(n+2)!\,(n^2-1)}\left\|g^{(n+1)}\right\|_\infty, \qquad (8)
\end{aligned}$$

which combined with identity (7) lead to first estimation in (3).

Let consider now that either n is an odd integer and $\frac{\beta}{\alpha} \in \left[\frac{1}{n}, n\right]$ which means $(n\alpha-\beta)(n\beta-\alpha) \geq 0$, or n is even and $\frac{\beta}{\alpha} \notin \left[\frac{1}{n}, n\right]$ $(a \neq 0)$, that is $(n\alpha-\beta)(n\alpha-\beta) \leq 0$. Then for $g(x) = x^{n+1}$ we find out easily that inequality (8) holds as an equality. That means, using the identity (7) as well, that the first inequality in (3) is sharp.

Now, if $g^{(n+1)} \in L_p[a,b]$, $1 < p < \infty$, then, by using Hölder's inequality, we have

$$\begin{aligned} & \frac{1}{(n+1)!(b-a)} \left| \int_a^b \int_a^b K_n(u,x) g^{(n+1)}(u) \, du dx \right| \\ \leq \; & \frac{1}{(n+1)!(b-a)} \left(\int_a^b \int_a^b \left| g^{(n+1)}(u) \right|^p du dx \right)^{\frac{1}{p}} \left(\int_a^b \int_a^b |K_n(u,x)|^q \, du dx \right)^{\frac{1}{q}} \\ = \; & \frac{\left(|n\alpha-\beta|^q + |n\beta-\alpha|^q\right)^{\frac{1}{q}} (b-a)^{n+\frac{1}{q}}}{(n-1)(n+1)! \left[(nq+1)(nq+2)\right]^{\frac{1}{q}}} \left\| g^{(n+1)} \right\|_p , \end{aligned}$$

which combined with the identity (7) gives the second inequality in (3).

Finally, let $g^{(n+1)} \in L_1[a,b]$. Then,

$$\begin{aligned} & \frac{1}{(n+1)!(b-a)} \left| \int_a^b \int_a^b K_n(u,x) g^{(n+1)}(u) \, du dx \right| \\ \leq \; & \frac{1}{(n+1)!(b-a)} \sup_{(u,x)\in[a,b]\times[a,b]} |K_n(u,x)| \int_a^b \int_a^b \left| g^{(n+1)}(u) \right| du dx \\ = \; & \frac{(b-a)^{n-1}}{(n-1)(n+1)!} \left\| g^{(n+1)} \right\|_1 \max\{|n\alpha-\beta|, |n\beta-\alpha|\}, \end{aligned}$$

which combined with (7) leads to third inequality in (3).

REMARK 1. If we apply the first inequality in (3) for $\alpha = \frac{n\rho-(-1)^n}{n+1}$, $\beta = \frac{\rho-(-1)^n n}{n+1}$, we get inequality (2). Therefore inequality (3) can be regarded as a generalization of (2).

If we apply the first inequality in (3) for $n = 2$ we have the following result:

COROLLARY 1. Let $g : [a,b] \to \mathbf{R}$ be a three time differentiable function on (a,b) with g''' bounded on (a,b). Then for all $\alpha, \beta \in \mathbf{R}$ we have the inequality

$$\begin{aligned} & \left| \frac{\alpha+\beta}{b-a} \int_a^b g(x) \, dx - (\alpha g(a) + \beta g(b)) + ((2\alpha-\beta) g'(a) - (2\beta-\alpha) g'(b)) \frac{(b-a)}{6} \right| \\ \leq \; & \frac{(|2\alpha-\beta| + |2\beta-\alpha|)(b-a)^3}{72} \left\| g''' \right\|_\infty . \end{aligned} \tag{9}$$

If $\alpha = 0$ or $\frac{\beta}{\alpha} \notin \left[\frac{1}{2}, 2\right]$, inequality (9) is sharp.

PROPOSITION 1. Let g be as in Corollary 1. Then,

$$\begin{aligned} & \left| g'(a) \int_a^b \left(g(x) - \frac{g(a)+2g(b)}{3} \right) dx + g'(b) \int_a^b \left(g(x) - \frac{2g(a)+g(b)}{3} \right) dx \right| \\ \leq \; & \frac{(|g'(a)| + |g'(b)|)(b-a)^4}{72} \left\| g''' \right\|_\infty , \end{aligned} \tag{10}$$

and inequality (10) is sharp.

PROOF. Applying (9) for $\alpha = \frac{2}{3}g'(b) + \frac{1}{3}g'(a)$ and $\beta = \frac{1}{3}g'(b) + \frac{2}{3}g'(a)$, we get (10). Further, an easy calculation yields that for $g(x) = (x-a)^3$ the equality in (10) holds. Therefore inequality (10) is sharp.

COROLLARY 2. Let g be as in Corollary 1. If $g'(a) = 0$ and $g'(b) \neq 0$, then the following holds

$$\left| \frac{1}{b-a} \int_a^b g(x)\,dx - \frac{2g(a) + g(b)}{3} \right| \leq \frac{(b-a)^3}{72} \left\| g''' \right\|_\infty, \tag{11}$$

while if $g'(a) \neq 0$ and $g'(b) = 0$, we have the inequality

$$\left| \frac{1}{b-a} \int_a^b g(x)\,dx - \frac{g(a) + 2g(b)}{3} \right| \leq \frac{(b-a)^3}{72} \left\| g''' \right\|_\infty. \tag{12}$$

Both inequalities are sharp.

PROOF. The above inequalities result in as an easy application from (10). In the case when $g(x) = (x-a)^3$ and $g(x) = (b-x)^3$ the equalities in (11) and (12) hold as well. Therefore the inequalities (11) and (12) are sharp.

3 An Application for Taylor's Remainder

As usual, $R_n(f; x_0, x)$ denotes the remainder in Taylor's formula, that is,

$$R_n(f; x_0, x) = f(x) - \sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x-x_0)^k}{k!}.$$

THEOREM2 . Let $I \subset \mathbf{R}$ be an open interval, and $x_0 \in I$. If $f : I \to \mathbf{R}$ is a $(n+2)$ −time differentiable function with $f^{(n+2)}$ integrable on I, then for all $x \in I$,

$$\begin{aligned} &\left| R_n(f; x_0, x) - \frac{x - x_0}{n+1} R_{n-1}(f'; x_0, x) \right| \\ \leq &\begin{cases} \frac{|x-x_0|^{n+2}}{(n+2)!(n+1)} \left\| f^{(n+2)} \right\|_\infty & \text{if } f^{(n+2)} \in L_\infty[\{x_0, x\}] \\ \frac{|x-x_0|^{n+1+\frac{1}{q}}}{(n+1)![(nq+1)(nq+2)]^{\frac{1}{q}}} \left\| f^{(n+2)} \right\|_p & \text{if } f^{(n+2)} \in L_p[\{x_0, x\}],\ 1 < p < \infty \\ \frac{|x-x_0|^n}{(n+1)!} \left\| f^{(n+2)} \right\|_1 & \text{if } f^{(n+2)} \in L_1[\{x_0, x\}] \end{cases}, \end{aligned}$$

where $[\{x_0, x\}]$ denotes the closed interval $[\min\{x, x_0\}, \max\{x, x_0\}]$. The first inequality is sharp.

PROOF. We distinguish two cases:

First case: Let $x \geq x_0$. Then applying inequalities (3) for $\alpha = \frac{1}{n+1}$, $\beta = \frac{n}{n+1}$, $g(x) = f'(x)$, $a = x$, $b = x_0$, and multiplying the result by $(x - x_0)$, we get,

$$\left| f(x) - f(x_0) - \frac{(f'(x) + nf'(x_0))}{n+1}(x - x_0) - \sum_{k=1}^{n-1} \frac{n-k}{n+1} f^{(k+1)}(x_0) \frac{(x-x_0)^{k+1}}{(k+1)!} \right|$$
$$\leq \begin{cases} \frac{(x-x_0)^{n+2}}{(n+2)!(n+1)} \left\| f^{(n+2)} \right\|_\infty & \text{if } f^{(n+2)} \in L_\infty[\{x_0, x\}] \\ \frac{(x-x_0)^{n+1+\frac{1}{q}}}{(n+1)![(nq+1)(nq+2)]^{\frac{1}{q}}} \left\| f^{(n+2)} \right\|_p & \text{if } f^{(n+2)} \in L_p[\{x_0, x\}], \ 1 < p < \infty \\ \frac{(x-x_0)^n}{(n+1)!} \left\| f^{(n+2)} \right\|_1 & \text{if } f^{(n+2)} \in L_1[\{x_0, x\}]. \end{cases} \quad (13)$$

Now, if we replace k by $k-1$, we have

$$\begin{aligned} & f(x) - f(x_0) - \frac{(f'(x) + nf'(x_0))}{n+1}(x - x_0) - \sum_{k=1}^{n-1} \frac{n-k}{n+1} f^{(k+1)}(x_0) \frac{(x-x_0)^{k+1}}{(k+1)!} \\ = \ & f(x) - f(x_0) - \frac{(f'(x) + nf'(x_0))}{n+1}(x - x_0) - \sum_{k=2}^{n} \frac{n+1-k}{n+1} f^{(k)}(x_0) \frac{(x-x_0)^{k}}{k!} \\ = \ & f(x) - f(x_0) - f'(x_0)(x - x_0) - \sum_{k=2}^{n} f^{(k)}(x_0) \frac{(x-x_0)^{k}}{k!} \\ & - \frac{x - x_0}{n+1} \left(f'(x) - f'(x_0) - \sum_{k=2}^{n} f^{(k)}(x_0) \frac{(x-x_0)^{k-1}}{(k-1)!} \right) \\ = \ & f(x) - \sum_{k=0}^{n} f^{(k)}(x_0) \frac{(x-x_0)^{k}}{k!} - \frac{x - x_0}{n+1} \left(f'(x) - \sum_{k=0}^{n-1} f^{(k+1)}(x_0) \frac{(x-x_0)^{k}}{k!} \right). \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} & f(x) - f(x_0) - \frac{(f'(x) + nf'(x_0))}{n+1}(x - x_0) - \sum_{k=1}^{n-1} \frac{n-k}{n+1} f^{(k+1)}(x_0) \frac{(x-x_0)^{k+1}}{(k+1)!} \\ = \ & R_n(f; x_0, x) - \frac{x - x_0}{n+1} R_{n-1}(f';, x_0, x). \end{aligned}$$

Finally, using this latter equality in (13) we get the desired result.

Second case: Let $x < x_0$. Applying (3) for $\alpha = \frac{n}{n+1}$, $\beta = \frac{1}{n+1}$, $g(x) = f'(x)$, $a = x_0$, $b = x$ and working in a way similar to the first case, we get the desired result as well. Finally, for $f(x) = x^{n+2}$, we readily calculate:

$$R_n(f; x_0, x) = (n+2)(x - x_0)^{n+1} x_0 + (x - x_0)^{n+2},$$

and

$$\begin{aligned} R_{n-1}(f';x_0,x) &= (n+2)\,R_{n-1}\left(x^{n+1};x_0,x\right) \\ &= (n+2)(n+1)(x-x_0)^n x_0 + (n+2)(x-x_0)^{n+1}. \end{aligned}$$

So we finally have

$$\left| R_n(f;x_0,x) - \frac{x-x_0}{n+1} R_{n-1}(f';x_0,x) \right| = \frac{1}{n+1}\,|x-x_0|^{n+2}.$$

On the other hand the following equality holds

$$\frac{|x-x_0|^{n+2}}{(n+2)!\,(n+1)} \left\| f^{(n+2)} \right\|_\infty = \frac{1}{n+1}\,|x-x_0|^{n+2}.$$

Combining those last two equalities we conclude that for $f(x) = x^{n+2}$ the equality in the first estimation of Theorem 2 holds. Consequently this inequality is sharp.

EXAMPLE 1. If we apply the first inequality of Theorem 2 for $f(x) = e^x$ and $x_0 = 0$, after some straightforward algebra it follows that

$$\left| e^x - \frac{1}{n+1-x} \sum_{k=0}^{n} (n+1-k) \frac{x^k}{k!} \right| \le \frac{x^{n+2}}{(n+2)!\,(n+1-x)} e^{\frac{x+|x|}{2}}.$$

EXAMPLE 2. If we apply the first inequality of Theorem 2 for $f(x) = \ln x,\ x, x_0 > 0$, and taking into account that,

$$R_n(f;x_0,x) = \ln x - \ln x_0 - \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-x_0)^k}{k x_0^k},$$

and

$$\begin{aligned} R_{n-1}(f';x_0,x) &= \frac{1}{x} - \frac{1}{x_0} \sum_{k=0}^{n-1} \frac{(x_0-x)^k}{x_0^k} \\ &= \frac{1}{x} - \frac{1}{x_0} \frac{1 - \frac{(x_0-x)^n}{x_0^n}}{1 - \frac{x_0-x}{x_0}} = \frac{(x_0-x)^n}{x x_0^n}, \end{aligned}$$

we get,

$$\begin{aligned} &\left| \ln x - \ln x_0 - \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-x_0)^k}{k x_0^k} + \frac{(x_0-x)^{n+1}}{(n+1)\, x x_0^n} \right| \\ \le\ & \frac{1}{(n+2)(n+1)} \frac{|x-x_0|^{n+2}}{(\min\{x,x_0\})^{n+2}}. \end{aligned}$$

4 Applications for Expectation and Variance

Many authors using trapezoid and Ostrowski type inequalities have produced estimations for the expectation and variance of a random variable X defined on a finite interval. For example in [4] we can see some related inequalities using an Ostrowski type inequality.

We want use some inequalities of the second section to obtain new inequalities for the expectation and the variance of a random variable X defined on a finite interval. Some similar results are presented in [1].

Let $f : [a, b] \to (0, \infty)$ be a differentiable function on (a, b) with f' bounded on (a, b). Assume that f is a probability density function of a random variable X, that is $\int_a^b f(x)\,dx = 1$. Denote by μ and σ^2 respectively the expectation and the variance of X.

PROPOSITION 2. The following inequalities hold,

$$\left|\sigma^2 + (b-\mu)\left(\frac{2a+b}{3} - \mu\right)\right| \le \frac{(b-a)^4}{36}\left\|f'\right\|_\infty, \tag{14}$$

and

$$\left|\sigma^2 + (a-\mu)\left(\frac{2b+a}{3} - \mu\right)\right| \le \frac{(b-a)^4}{36}\left\|f'\right\|_\infty. \tag{15}$$

Both inequalities are sharp.

PROOF. Let $g : [a, b] \to R_+$ be the function given by $g(x) = \int_a^x \int_a^u f(t)\,dtdu$. Then we have, that $g'(a) = 0,\ g'(b) = 1 \neq 0$. Now, if we apply (11) for g, and taking into account that we have

$$g(a) = 0,$$

$$g(b) = \int_a^b (u-b)' \int_a^u f(t)\,dtdu = \int_a^b (b-u) f(u)\,du = b - \mu,$$

and

$$\begin{aligned}\int_a^b g(x)\,dx &= \int_a^b (x-b)' \int_a^x \int_a^u f(t)\,dtdudx = -\int_a^b (x-b)\int_a^x f(t)\,dtdx \\ &= -\frac{1}{2}\int_a^b \left((x-b)^2\right)' \int_a^x f(t)\,dtdx = \frac{1}{2}\int_a^b (x-b)^2 f(x)\,dx \\ &= \frac{1}{2}\left(\sigma^2 + (b-\mu)^2\right),\end{aligned}$$

we directly get (14). An easy calculation yields that for $f(x) = \frac{2(x-a)}{(b-a)^2}$, (14) holds as an equality. So inequality (14) is sharp.

Let now $g(x) = \int_b^x \int_b^u f(t)\,dtdu,\ x \in [a, b]$. Then, in a similar way as above we could state that

$$\begin{aligned} g(b) &= g'(b) = 0,\ g'(a) = -1,\ g(a) = \mu - a, \\ \int_a^b g(x)\,dx &= \frac{1}{2}\left(\sigma^2 + (\mu - a)^2\right). \end{aligned}$$

So, we can apply (12) to obtain (15), and is easy to verify that for $f(x) = \frac{2(b-x)}{(b-a)^2}$ the equality in (15) holds.

Now, applying Corollary 1 first for $g(x) = \int_a^x \int_a^u f(t)\,dtdu$ and then for $g(x) = \int_b^x \int_b^u f(t)\,dtdu$, in similar way as above, we can prove the following proposition.

PROPOSITION 3. For all $\alpha, \beta \in \mathbf{R}$, we have the inequalities,

$$\begin{aligned} &\left|3(\alpha+\beta)\left(\sigma^2+(b-\mu)^2\right)-6\beta(b-\mu)(b-a)-(2\beta-\alpha)(b-a)^2\right| \\ \leq\ & \frac{(|2\alpha-\beta|+|2\beta-\alpha|)(b-a)^4}{12}\left\|f'\right\|_\infty, \end{aligned} \tag{16}$$

and

$$\begin{aligned} &\left|3(\alpha+\beta)\left(\sigma^2+(\mu-a)^2\right)-6\alpha(\mu-a)(b-a)-(2\alpha-\beta)(b-a)^2\right| \\ \leq\ & \frac{(|2\alpha-\beta|+|2\beta-\alpha|)(b-a)^4}{12}\left\|f'\right\|_\infty. \end{aligned} \tag{17}$$

REMARK 2. Setting $\beta = -\alpha$ in inequalities (16) and (17) we find that

$$\left|\mu - \frac{3b-a}{2}\right| \leq \frac{(b-a)^3}{12}\left\|f'\right\|_\infty,$$

and

$$\left|\mu - \frac{3a-b}{2}\right| \leq \frac{(b-a)^3}{12}\left\|f'\right\|_\infty.$$

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