Applied Mathematics E-Notes, 9(2009), 297-299 © ISSN 1607-2510
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Counting Primes In The Quadratic Intervals*

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Received 29 November 2008

Abstract

In direction of the classical conjecture of the existence of prime numbers in all quadratic intervals $(n^2,(n+1)^2)$, we show that there are infinity many positive integer values of n such that this interval contains more than $\frac{n}{1+\log n}$ primes.

For $\alpha \geq 1$, we set $\Delta\pi^{[\alpha]}(n) = \pi\big((n+1)^\alpha\big) - \pi(n^\alpha)$, where $\pi(x)$ is the number of primes not exceeding x. A classical conjecture in Number Theory asserts that all quadratic intervals $(n^2,(n+1)^2)$ contain primes, i.e., the inequality $\Delta\pi^{[2]}(n) \geq 1$ holds for all n. In this short note, related by this conjecture, we show that

$$\limsup_{n\to\infty} \frac{\Delta\pi^{[2]}(n)}{n/\log n} \geq 1.$$

To do this, we use the following sharp bounds [1] for the function $\pi(x)$:

$$L(x) := \frac{x}{\log x}\Big(1+\frac{1}{\log x}+\frac{1.8}{\log^2 x}\Big) \leq \pi(x) \qquad (x \geq 32299), \tag{1}$$

and

$$\pi(x) \leq U(x) := \frac{x}{\log x}\Big(1+\frac{1}{\log x}+\frac{2.51}{\log^2 x}\Big) \qquad (x \geq 355991). \tag{2}$$

More precisely, we prove:

THEOREM 1. For infinity many positive integer values of n, the following inequality holds

$$\frac{n}{1+\log n} \leq \Delta\pi^{[2]}(n).$$

PROOF. Let $x = n^2$ in (1). Then for $n \geq 180 = \lceil \sqrt{32299}\, \rceil$ we obtain

$$\frac{n^2}{2\log n}\left(1+\frac{1}{2\log n}+\frac{9}{20\log^2 n}\right) < \pi(n^2).$$

*Mathematics Subject Classifications: 11A41, 11N05.

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Also, it is clear that for every $n \geq 2$, we have

$$\frac{n^2}{2\log n} + 4 - \frac{9}{\log 9} - \sum_{k=3}^{n-1} \frac{\log^2 k}{\log\log k} < \frac{n^2}{2\log n}\left(1 + \frac{1}{2\log n} + \frac{9}{20\log^2 n}\right).$$

We combine these two inequalities to get the following inequality

$$\frac{n^2}{2\log n} + 4 - \frac{9}{\log 9} - \sum_{k=3}^{n-1} \frac{\log^2 k}{\log\log k} < \pi(n^2) \qquad (n \geq 180),$$

and we rewrite this as follows

$$\frac{1}{2}\left(\frac{n^2}{\log n} - \frac{3^2}{\log 3}\right) - \sum_{k=3}^{n-1} \frac{\log^2 k}{\log\log k} < \pi(n^2) - \pi(3^2).$$

This inequality yields that

$$\sum_{k=3}^{n-1} \left\lfloor \frac{1}{2}\left(\frac{(k+1)^2}{\log(k+1)} - \frac{k^2}{\log k}\right) - \frac{\log^2 k}{\log\log k} \right\rfloor < \sum_{k=3}^{n-1} \pi\big((k+1)^2\big) - \pi(k^2) \qquad (n \geq 180).$$

Now, we note that terms under summations on both sides, are non-negative integers, and this asserts that the inequality

$$\left\lfloor \frac{1}{2}\left(\frac{(n+1)^2}{\log(n+1)} - \frac{n^2}{\log n}\right) - \frac{\log^2 n}{\log\log n} \right\rfloor \leq \pi\big((n+1)^2\big) - \pi(n^2),$$

holds for infinity many positive integer values of n. Since for $n \geq 7413$ the left hand side of the last inequality is greater than $\frac{n}{1+\log n}$, we obtain the result. This completes the proof.

NOTE 1. The following stronger version of the above result has been checked by computer for $3 \leq n \leq 10000$:

$$-\frac{\log^2 n}{\log\log n} - 1 < \Delta\pi^{[2]}(n) - \frac{1}{2}\left(\frac{(n+1)^2}{\log(n+1)} - \frac{n^2}{\log n}\right) < \log^2 n \log\log n.$$

This may hold for all values of n, "this is a conjecture".

NOTE 2. Let

$$g(n) = \#\big\{t \mid t \in \mathbb{N},\ t \leq n,\ \big(t^2, (t+1)^2\big) \text{ contains a prime}\big\}.$$

Clearly, $\lim_{n\to\infty} g(n) = \infty$ and $g(n) \leq n$. A lower bound for $g(n)$ is $g(n) \geq M(n)$, where

$$M(n) = \max_m \left\{ \sum_{k=597}^{n} \left\lfloor \frac{1}{2}\left(\frac{(k+1)^2}{\log(k+1)} - \frac{k^2}{\log k}\right) - \frac{\log^2 k}{\log\log k} \right\rfloor \leq \sum_{k=m}^{n} U\big((k+1)^2\big) - L(k^2) \right\}.$$

This holds for every $n \geq 597$, and obtained by considering Theorem 1. As $n \to \infty$, we have

$$M(n) = O(n).$$

Finally, we guess that for every $\epsilon > 0$ there exists $n_\epsilon \in \mathbb{N}$ such that for all $n > n_\epsilon$ we have $M(n) > (1 - \epsilon)n$.

NOTE 3. We end this note by introducing a question. What is the value of the following quantity $\inf \Big\{\alpha : \ \Delta\pi^{[\alpha]}(n) \geq 1 \text{ holds for all } n \in \mathbb{N}\Big\}$?

Acknowledgment. I express my thanks to Ejkova Ekaterina (Katya) for her useful comments on the English writing of this note.

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