Applied Mathematics E-Notes, 9(2009), 289-296 © ISSN 1607-2510
Available free at mirror sites of http://www.math.nthu.edu.tw/∼amen/

Perron Complements Of Diagonally Dominant Matrices And H-Matrices*

Li Zeng[†], Ming Xiao[‡], Ting-Zhu Huang[§]

Received 27 November 2008

Abstract

In this paper, we consider properties of the Perron complements of diagonally dominant matrices and H-matrices.

1 Introduction

Let $A = (a_{ij})$ be an $n \times n$ matrix, and recall that A is (row) diagonally dominant if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|, \; i = 1, 2, ..., n. \tag{1}$$

A is further said to be strictly diagonally dominant if all the strict inequalities in (1) hold. Obviously the principal submatrices of strictly diagonally matrices are strictly diagonally dominant and thus A is nonsingular.

For $A \in Z = \{(a_{ij}) \in R^{n,n} : a_{ij} \leq 0, i \neq j\}$, if $A = aI - B, B \geq 0, a > \rho(B)$, then A is called an M-matrix. The comparison matrix $\mu(A) = (\mu_{ij})$ is defined by

$$\mu_{ij} = \begin{cases} -|a_{ij}| & i \neq j, \\ |a_{ij}| & i = j. \end{cases}$$

$A \in C^{n,n}$ is called an H-matrix if $\mu(A)$ is an M-matrix. If there exists a positive diagonal matrix $D = \mathrm{diag}(d_1, ..., d_n)$ such that $D^{-1}AD$ is strictly diagonally dominant, we call A a generalized diagonally dominant matrix. It is well-known that A is an H-matrix is equivalent to A is generalized diagonally dominant.

The empty set is denoted by ϕ. Let α, β be nonempty ordered subsets of $\langle n \rangle := \{1, 2, ..., n\}$, both consisting of strictly increasing integers. By $A(\alpha, \beta)$ we shall denote the submatrix of A lying in rows indexed by α and columns indexed by β. If, in addition, $\alpha = \beta$, then the principal submatrix $A(\alpha, \alpha)$ is abbreviated to $A(\alpha)$.

---

*Mathematics Subject Classifications: 15A48

[†]College of Computer Science and Technology, Southwest University for Nationalities, Chengdu, Sichuan, 610041, P. R. China

[‡]College of Computer Science and Technology, Southwest University for Nationalities, Chengdu, Sichuan, 610041, P. R. China

[§]School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu, 610054, P. R. China

Suppose that $\alpha \subset \langle n \rangle$. If $A(\alpha)$ is nonsingular, then the Schur complement of $A(\alpha)$ in A is given by

$$S(A/A(\alpha)) = A(\beta) - A(\beta,\alpha)[A(\alpha)]^{-1} A(\alpha,\beta), \tag{2}$$

where $\beta = \langle n \rangle \setminus \alpha$. A well-known result due to Carlson and Markham [1] states that the Schur complements of strictly diagonally dominant matrices are diagonally dominant.

For an $n \times n$ nonnegative and irreducible matrix A, Meyer [2,3] introduced the notion of the Perron complement. Again, let $\alpha \subset \langle n \rangle$ and $\beta = \langle n \rangle \setminus \alpha$. Then the Perron complement of $A(\alpha)$ in A is given by

$$P(A/A(\alpha)) = A(\beta) + A(\beta,\alpha)[\rho(A) I - A(\alpha)]^{-1} A(\alpha,\beta), \tag{3}$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix. Recall that as A is irreducible, $\rho(A) > \rho(A(\alpha))$, so that the expression on the right-hand side of (3) is well defined, and we observe that $\rho(A) I - A(\alpha)$ is an M-matrix and thus $(\rho(A) I - A(\alpha))^{-1} \geq 0$. Meyer [2,3] has derived several interesting and useful properties of $P(A/A(\alpha))$, such as $P(A/A(\alpha))$ is also nonnegative and irreducible, and $\rho(P(A/A(\alpha))) = \rho(A)$. In addition, the Perron complements of inverse M-matrices [4] have also been studied.

For any $\alpha \subset \langle n \rangle$ and for any $t \geq \rho(A)$, let the extended Perron complement at t be the matrix

$$P_t(A/A(\alpha)) = A(\beta) + A(\beta,\alpha)[tI - A(\alpha)]^{-1} A(\alpha,\beta), \tag{4}$$

which is also well defined since $t \geq \rho(A) > \rho(A(\alpha))$.

In this paper, we shall show, in Section 2, that the Perron complement of a diagonally dominant and nonnegative irreducible matrix ,

$$P(A/A(\alpha)) = A(\beta) + A(\beta,\alpha)[\rho(A) I - A(\alpha)]^{-1} A(\alpha,\beta),$$

is diagonally dominant only if $\rho(A) \geq \max_{i\in\alpha} \sum_{j=1}^{n} |a_{ij}|$. In Section 3, we shall show a similar result for H-matrices.

2 Perron Complements of Diagonally Dominant Matrices

First recall the following result proved in [2].

LEMMA 2.1 ([2]). If A is a nonnegative irreducible matrix with spectral radius $\rho(A)$, and let $\alpha \subset \langle n \rangle$, $\alpha \neq \phi$ and $\beta = \langle n \rangle \setminus \alpha$. Then the Perron complement

$$P(A/A(\alpha)) = A(\beta) + A(\beta,\alpha)[\rho(A) I - A(\alpha)]^{-1} A(\alpha,\beta)$$

is also a nonnegative irreducible matrix with spectral radius $\rho(A)$.

We are now in a position to state the main result of the paper on the Perron complements of diagonally dominant matrices.

THEOREM 2.2. Let A be an $n\times n$ diagonally dominant and nonnegative irreducible matrix with spectral radius $\rho(A)$, and let $\alpha \subset \langle n\rangle$, $\alpha \neq \phi$ and $\beta = \langle n\rangle \setminus \alpha$. Then, for $\rho(A) \geq \max_{i\in\alpha} \sum\limits_{j=1}^{n} |a_{ij}|$,

$$P(A/A(\alpha)) = A(\beta) + A(\beta,\alpha)\left[\rho(A)I - A(\alpha)\right]^{-1} A(\alpha,\beta)$$

is a diagonally dominant and nonnegative irreducible matrix.

PROOF. Let $\alpha = \{i_1, i_2, ..., i_k\}$ and $\beta = \{j_1, j_2, ..., j_l\}$, where $k + l = n$. Denote $|A| = (|a_{ij}|)$. Since A is a diagonally dominant matrix, we have, for any $i \in \langle n\rangle$,

$$|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$$

or

$$|a_{j_t j_t}| \geq \sum_{s=1,\neq t}^{l} |a_{j_t j_s}| + \sum_{s=1}^{k} |a_{j_t i_s}|, \tag{5}$$

where $j_t, j_s \in \beta, i_s \in \alpha$. Note that A is an irreducible and nonnegative matrix, then $\rho(A) > \rho(A(\alpha))$, so that $\rho(A)I - A(\alpha)$ is an M-matrix. Then we have

$$(\rho(A)I - A(\alpha))^{-1} \geq 0 \text{ and } a_{ij} \geq 0. \tag{6}$$

By $\rho(A) \geq \max_{i\in\alpha} \sum\limits_{j=1}^{n} |a_{ij}|$, we have

$$\rho(A) \geq \max_{i\in\alpha}\left(\sum_{j\in\beta}|a_{ij}| + \sum_{j\in\alpha}|a_{ij}|\right) = \max_{i_v\in\alpha}\sum_{t=1}^{l} a_{i_v j_t} + \max_{i_v\in\alpha}\sum_{t=1}^{k} a_{i_v i_t}. \tag{7}$$

If $\max_{i_v\in\alpha} \sum\limits_{t=1}^{l} a_{i_v j_t} = 0$, then $P(A/A(\alpha)) = A(\beta)$. Thus, the matrix $P(A/A(\alpha))$ is diagonally dominant. If $\max_{i_v\in\alpha} \sum\limits_{t=1}^{l} a_{i_v j_t} > 0$, then, by (7), we have

$$0 < \max_{i_v\in\alpha}\sum_{t=1}^{l} a_{i_v j_t} \leq \rho(A) - \max_{i_v\in\alpha}\sum_{t=1}^{k} a_{i_v i_t}. \tag{8}$$

Thus

$$\max_{i_v\in\alpha}\frac{\sum\limits_{t=1}^{l} a_{i_v j_t}}{\rho(A) - \sum\limits_{t=1}^{k} a_{i_v i_t}} \leq 1. \tag{9}$$

Denote

$$x = (\rho(A)I - A(\alpha))^{-1}\left(\sum_{s=1}^{l} a_{i_1 j_s}, ..., \sum_{s=1}^{l} a_{i_k j_s}\right)^{\dagger} \tag{10}$$

or

$$\left(\rho\left(A\right)I-A\left(\alpha\right)\right)x=\left(\sum_{s=1}^{l}a_{i_1j_s},...,\sum_{s=1}^{l}a_{i_kj_s}\right)^{\dagger}.$$

Letting $x_v=\max\{x_1,x_2,\cdots,x_k\}$, where x_i is the i-th component of x, we obtain

$$\begin{aligned}\sum_{s=1}^{l}a_{i_vj_s} &= \left(\rho\left(A\right)-a_{i_vi_v}\right)x_v+\sum_{t=1,\neq v}^{k}(-a_{i_vi_t})x_t\\ &\geq \left(\rho\left(A\right)-a_{i_vi_v}+\sum_{t=1,\neq v}^{k}(-a_{i_vi_t})\right)x_v\\ &= \left(\rho\left(A\right)-\sum_{t=1}^{k}a_{i_vi_t}\right)x_v.\end{aligned}$$

By (8), we have

$$x_v\leq\frac{\sum_{t=1}^{l}a_{i_vj_t}}{\rho\left(A\right)-\sum_{t=1}^{k}a_{i_vi_t}}\leq\max\nolimits_{i_v\in\alpha}\frac{\sum_{t=1}^{l}a_{i_vj_t}}{\rho\left(A\right)-\sum_{t=1}^{k}a_{i_vi_t}}.$$

By (9), we have

$$x_v\leq 1. \tag{11}$$

Denote the (t,s)-entry of $P\left(A/A\left(\alpha\right)\right)$ by $\left(a'_{j_tj_s}\right)$. Then, for $t=1,2,...,l$, we have

$$\begin{aligned}&\left|a'_{j_tj_t}\right|-\sum_{s=1,\neq t}^{l}\left|a'_{j_tj_s}\right|\\ =&\left|a_{j_tj_t}+\left(a_{j_ti_1},...,a_{j_ti_k}\right)\left(\rho\left(A\right)I-A\left(\alpha\right)\right)^{-1}\begin{pmatrix}a_{i_1j_t}\\ \vdots\\ a_{i_kj_t}\end{pmatrix}\right|\\ &-\sum_{s=1,\neq t}^{l}\left|a_{j_tj_s}+\left(a_{j_ti_1},...,a_{j_ti_k}\right)\left(\rho\left(A\right)I-A\left(\alpha\right)\right)^{-1}\begin{pmatrix}a_{i_1j_s}\\ \vdots\\ a_{i_kj_s}\end{pmatrix}\right|,\end{aligned}$$

so that

$$\begin{aligned}
& \left|a'_{j_tj_t}\right| - \sum_{s=1,\neq t}^{l} \left|a'_{j_tj_s}\right| \\
\geq \ & \left[\left|a_{j_tj_t}\right| - \left(\left|a_{j_ti_1}\right|, ..., \left|a_{j_ti_k}\right|\right) \left|\left(\rho\left(A\right)I - A\left(\alpha\right)\right)^{-1}\right| \begin{pmatrix} \left|a_{i_1j_t}\right| \\ \vdots \\ \left|a_{i_kj_t}\right| \end{pmatrix} \right] \\
& - \sum_{s=1,\neq t}^{l} \left[\left|a_{j_tj_s}\right| + \left(\left|a_{j_ti_1}\right|, ..., \left|a_{j_ti_k}\right|\right) \left|\left(\rho\left(A\right)I - A\left(\alpha\right)\right)^{-1}\right| \begin{pmatrix} \left|a_{i_1j_s}\right| \\ \vdots \\ \left|a_{i_kj_s}\right| \end{pmatrix} \right] \\
= \ & a_{j_tj_t} - \sum_{s=1,\neq t}^{l} a_{j_tj_s} - \left(a_{j_ti_1}, ..., a_{j_ti_k}\right)\left(\rho\left(A\right)I - A\left(\alpha\right)\right)^{-1} \begin{pmatrix} \sum_{s=1}^{l} a_{i_1j_s} \\ \vdots \\ \sum_{s=1}^{l} a_{i_kj_s} \end{pmatrix} \\
\geq \ & a_{j_tj_t} - \sum_{s=1,\neq t}^{l} a_{j_tj_s} - \left(a_{j_ti_1}, ..., a_{j_ti_k}\right) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \\
= \ & \left|a_{j_tj_t}\right| - \sum_{s=1,\neq t}^{l} \left|a_{j_tj_s}\right| - \sum_{s=1}^{k} \left|a_{j_ti_s}\right| \\
\geq \ & 0.
\end{aligned}$$

It follows that $P\left(A/A\left(\alpha\right)\right)$ is a diagonally dominant matrix. By Lemma 2.1, the matrix $P\left(A/A\left(\alpha\right)\right)$ is nonnegative irreducible. This completes the proof.

By Theorem 2.2, we have several immediate results about the extended Perron complements and the Perron complements of strictly diagonally dominant matrices.

COROLLARY 2.3. Let A be an $n \times n$ diagonally dominant and nonnegative irreducible matrix with spectral radius $\rho\left(A\right)$, and let $\alpha \subset \langle n \rangle, \alpha \neq \phi$ and $\beta = \langle n \rangle \setminus \alpha$. Then, for any $t \in \left[\rho\left(A\right), \infty\right)$ and $\rho\left(A\right) \geq \max_{i\in\alpha} \sum_{j=1}^{n} \left|a_{ij}\right|$,

$$P_t\left(A/A\left(\alpha\right)\right) = A\left(\beta\right) + A\left(\beta, \alpha\right)\left[tI - A\left(\alpha\right)\right]^{-1} A\left(\alpha, \beta\right)$$

is a diagonally dominant and nonnegative irreducible matrix.

COROLLARY 2.4. Let A be an $n \times n$ strictly diagonally dominant and nonnegative irreducible matrix with spectral radius $\rho\left(A\right)$, and let $\alpha \subset \langle n \rangle, \alpha \neq \phi$ and $\beta = \langle n \rangle \setminus \alpha$. Then, for any $t \in \left[\rho\left(A\right), \infty\right)$ and $\rho\left(A\right) \geq \max_{i\in\alpha} \sum_{j=1}^{n} \left|a_{ij}\right|$, $P\left(A/A\left(\alpha\right)\right)$ and $P_t\left(A/A\left(\alpha\right)\right)$ are strictly diagonally dominant and nonnegative irreducible matrices.

3 Perron Complements of H-matrices

In this section, we obtain a theorem of the Perron complements of H-matrices.

THEOREM 3.1. Let A be an $n \times n$ nonnegative irreducible H-matrix with spectral radius $\rho(A)$, and let $\alpha \subset \langle n \rangle, \alpha \neq \phi$ and $\beta = \langle n \rangle \backslash \alpha$. Then, for $\rho(A) \geq \max_{i\in\alpha} \sum_{j=1}^{n} |a_{ij}| \geq 2|a_{ii}|, i \in \alpha$,

$$P(A/A(\alpha)) = A(\beta) + A(\beta,\alpha)[\rho(A)I - A(\alpha)]^{-1} A(\alpha,\beta)$$

is a nonnegative irreducible H-matrix.

PROOF. Let $\alpha = \{i_1, i_2, ..., i_k\}$ and $\beta = \{j_1, j_2, ..., j_l\}$, where $k + l = n$. Since A is an H-matrix, then there exists a positive diagonal matrix

$$X = \operatorname{diag}(x_1, x_2, ..., x_n) > 0$$

such that $X^{-1}AX$ is a strictly diagonally dominant matrix, i.e.,

$$|a_{ii}| > \sum_{j\neq i} \frac{x_j}{x_i} |a_{ij}|, i \in \langle n \rangle.$$

Suppose that $B = (b_{ij}) = X^{-1}AX$, we have $\rho(B) = \rho(A)$ and B is a strictly diagonally dominant matrix. Since

$$\rho(A) \geq \max_{i\in\alpha} \sum_{j=1}^{n} |a_{ij}| \geq 2|a_{ii}|, i \in \alpha$$

and

$$|a_{ii}| > \sum_{j\neq i} \frac{x_j}{x_i} |a_{ij}|, i \in \langle n \rangle,$$

we have $\rho(B) = \rho(A) \geq \max_{i\in\alpha} \sum_{j=1}^{n} |b_{ij}|$. Then,by Corollary 2.4, $P(B/B(\alpha))$ is a strictly diagonally dominant matrix and

$$B = \begin{bmatrix} a_{11} & \frac{x_2}{x_1}a_{12} & \cdots & \frac{x_n}{x_1}a_{1n} \\ \frac{x_1}{x_2}a_{21} & a_{22} & \cdots & \frac{x_n}{x_2}a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{x_1}{x_n}a_{n1} & \frac{x_2}{x_n}a_{n2} & \cdots & a_{nn} \end{bmatrix}.$$

Let $D = \operatorname{diag}(x_{j_1}, x_{j_2}, ..., x_{j_l}) > 0$. Then,

$$
\begin{aligned}
P(B/B(\alpha)) &= B(\beta) + B(\beta,\alpha)[\rho(B)I - B(\alpha)]^{-1} B(\alpha,\beta) \\
&= \begin{bmatrix} b_{j_1j_1} & \cdots & b_{j_1j_l} \\ \cdots & \cdots & \cdots \\ b_{j_lj_1} & \cdots & b_{j_lj_l} \end{bmatrix} + \begin{bmatrix} b_{j_1i_1} & \cdots & b_{j_1i_k} \\ \cdots & \cdots & \cdots \\ b_{j_li_1} & \cdots & b_{j_li_k} \end{bmatrix} \\
&\quad \times \begin{bmatrix} \rho(B) - b_{i_1i_1} & \cdots & -b_{i_1i_k} \\ \cdots & \cdots & \cdots \\ -b_{i_ki_1} & \cdots & \rho(B) - b_{i_ki_k} \end{bmatrix} \begin{bmatrix} b_{i_1j_1} & \cdots & b_{i_1j_l} \\ \cdots & \cdots & \cdots \\ b_{i_kj_1} & \cdots & b_{i_kj_l} \end{bmatrix} \\
&= \begin{bmatrix} a_{j_1j_1} & \cdots & \frac{x_{j_l}}{x_{j_1}} a_{j_1j_l} \\ \cdots & \cdots & \cdots \\ \frac{x_{j_1}}{x_{j_l}} a_{j_lj_1} & \cdots & a_{j_lj_l} \end{bmatrix} + \begin{bmatrix} \frac{x_{i_1}}{x_{j_1}} a_{j_1i_1} & \cdots & \frac{x_{i_k}}{x_{j_1}} a_{j_1i_k} \\ \cdots & \cdots & \cdots \\ \frac{x_{i_1}}{x_{j_l}} a_{j_li_1} & \cdots & \frac{x_{i_k}}{x_{j_l}} a_{j_li_k} \end{bmatrix} \\
&\quad \times \begin{bmatrix} \rho(A) - a_{i_1i_1} & \cdots & -\frac{x_{i_k}}{x_{i_1}} a_{i_1i_k} \\ \cdots & \cdots & \cdots \\ -\frac{x_{i_1}}{x_{i_k}} a_{i_ki_1} & \cdots & \rho(A) - a_{i_ki_k} \end{bmatrix} \begin{bmatrix} \frac{x_{j_1}}{x_{i_1}} a_{i_1j_1} & \cdots & \frac{x_{j_l}}{x_{i_1}} a_{i_1j_l} \\ \cdots & \cdots & \cdots \\ \frac{x_{j_1}}{x_{i_k}} a_{i_kj_1} & \cdots & \frac{x_{j_l}}{x_{i_k}} a_{i_kj_l} \end{bmatrix} \\
&= \operatorname{diag}\left(\frac{1}{x_{j_1}}, ..., \frac{1}{x_{j_l}}\right) \begin{bmatrix} a_{j_1j_1} & \cdots & a_{j_1j_l} \\ \cdots & \cdots & \cdots \\ a_{j_lj_1} & \cdots & a_{j_lj_l} \end{bmatrix} \operatorname{diag}(x_{j_1}, ..., x_{j_l}) \\
&\quad + \operatorname{diag}\left(\frac{1}{x_{j_1}}, ..., \frac{1}{x_{j_l}}\right) \begin{bmatrix} a_{j_1i_1} & \cdots & a_{j_1i_k} \\ \cdots & \cdots & \cdots \\ a_{j_li_k} & \cdots & a_{j_li_k} \end{bmatrix} \operatorname{diag}(x_{i_1}, ..., x_{i_k}) \\
&\quad \times \operatorname{diag}\left(\frac{1}{x_{i_1}}, ..., \frac{1}{x_{i_k}}\right) \begin{bmatrix} \rho(A) - a_{i_1i_1} & \cdots & -a_{i_1i_k} \\ \cdots & \cdots & \cdots \\ -a_{i_ki_1} & \cdots & \rho(A) - a_{i_ki_k} \end{bmatrix} \operatorname{diag}(x_{i_1}, ..., x_{i_k}) \\
&\quad \times \operatorname{diag}\left(\frac{1}{x_{i_1}}, ..., \frac{1}{x_{i_k}}\right) \begin{bmatrix} a_{i_1j_1} & \cdots & a_{i_1j_l} \\ \cdots & \cdots & \cdots \\ a_{i_kj_1} & \cdots & a_{i_kj_l} \end{bmatrix} \operatorname{diag}(x_{j_1}, ..., x_{j_l}) \\
&= D^{-1} A(\beta) D + D^{-1} A(\beta,\alpha)[\rho(A)I - A(\alpha)]^{-1} A(\alpha,\beta) D \\
&= D^{-1} P(A/A(\alpha)) D.
\end{aligned}
$$

Note that the matrix

$$P(B/B(\alpha)) = D^{-1} P(A/A(\alpha)) D$$

is strictly diagonally dominant, then $P(A/A(\alpha))$ is an H-matrix. By Lemma 2.1, we have the matrix $P(A/A(\alpha))$ is nonnegative irreducible. This completes the proof.

4 Example

Let

$$A = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & 0 & 1 \\ 2 & 1 & 4 & 1 \\ 1 & 2 & 1 & 4 \end{pmatrix}.$$

Obviously, A is a diagonally dominant and nonnegative irreducible H-matrix. And,

$$\rho(A) = 6.3028 \geq \max_{i\in\alpha} \sum_{j=1}^{n} |a_{ij}| \geq 2\,|a_{ii}|\,, i \in \alpha, \alpha = \{1\} \text{ or } \{1,2\}.$$

Then,

$$P(A/A(\alpha)) = \begin{pmatrix} 3.3028 & 0.3028 & 1 \\ 1.6055 & 4.6055 & 1 \\ 2.3028 & 1.3028 & 4 \end{pmatrix},$$

where $\alpha = \{1\}$, is a diagonally dominant H-matrix. And,

$$P(A/A(\alpha)) = \begin{pmatrix} 4.7675 & 1.5351 \\ 1.5351 & 4.7675 \end{pmatrix},$$

where $\alpha = \{1,2\}$, is a diagonally dominant H-matrix.

References

[1] D. Carlson and T. Markham, Schur complements of diagonally dominant matrices, Czech. Math. J., 29(104)(1979), 246–251.

[2] C. D. Meyer, Uncoupling the Perron eigenvector problem, Linear Algebra Appl., 114/115(1989), 69–94.

[3] C. D. Meyer, Stochastic complementation uncoupling Markov chains and the theory of nearly reducible systems, SIAM Rev., 31(1989), 240–272.

[4] M. Neumann, Inverses of Perron complements of inverse M-matrices, Linear Algebra Appl., 313(2000), 163–171.