

Lie Symmetries Analysis Of The Shallow Water Equations*

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Abstract

In this work we consider an auxiliary system of the shallow water equations in which the first equation is written in a conserved form. So a five dimensional symmetry algebra is obtained. Similarity reductions are performed for each generator and non trivial exact solutions are given.

1 Introduction

The basic idea of Lie symmetry method is to reduce the order or the number of independent variables of differential equations under consideration as much as possible so that the integration can be done easily.

The basic shallow water equations on a flat bottom are written in the following form:

$$\begin{cases} h_t + uh_x + hu_x = 0, \\ u_t + uu_x + h_x = 0. \end{cases} \quad (1)$$

where $u = u(x, t)$ is the velocity and $h = h(x, t)$ the depth of the water. Here the indices denote the derivatives with respect to appropriate arguments. The system (1) is an important model of mathematical physics with applications in different fields as for example in engineering, hydraulics [5], topography, etc.

In applied sciences the system (1) describes new physical phenomena by including additional terms or with boundary conditions, for example: the circulation of bodies of water in coastal zone, nonlinear transformation of the swell, morphodynamic evolution of the coast, environmental protection (pollution at sea, follow-up of rejections in aquatic environment). In experimental approach many problems are treated by means of one dimensional models.

This work is organized as follows. In the next section, applying Lie's algorithm method to an auxiliary system associated to (1), we obtain Lie point symmetries admitted by the system (1). Commutation relations between generators are also given.

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In section 3, using the obtained Lie point symmetries we reduce the shallow water equations to a family of different systems from which it arises some nontrivial exact solutions.

2 The Symmetry Groups of (1)

As the first equation in (1) can be written in conserved form, i.e.,

$$\begin{cases} h_t + (uh)_x & = 0, \\ u_t + (\frac{u^2}{2} + h)_x & = 0, \end{cases} \quad (2)$$

then its associated auxiliary system is given by

$$\begin{cases} v_x & = h, \\ v_t & = -uh, \\ u_t + uu_x + h_x & = 0. \end{cases} \quad (3)$$

By V we mean a vector field and let $Pr^{(1)}V$ be the first prolongation of V . So if V is given locally by

$$V = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial t} + \varphi \frac{\partial}{\partial u} + H \frac{\partial}{\partial h} + G \frac{\partial}{\partial v}, \quad (4)$$

where ξ, η, φ, H and G are infinitesimal depending on the independent variables (x, t) , and the dependent variables u, h and potential v , then

$$Pr^{(1)}V = V + \varphi^x \frac{\partial}{\partial u_x} + \varphi^t \frac{\partial}{\partial u_t} + H^x \frac{\partial}{\partial h_x} + H^t \frac{\partial}{\partial h_t} + G^x \frac{\partial}{\partial v_x} + G^t \frac{\partial}{\partial v_t}. \quad (5)$$

Using the first prolongation formulae [1, 2, 3] and the system (3) we get

$$\begin{aligned} \varphi^x &= \varphi_x + h\varphi_v + (\varphi_u - \xi_x - h\xi_v + u\eta_x + uh\eta_v)u_x + (\varphi_h + \eta_x + h\eta_v)h_x \\ &+ (-\xi_h + \eta_u + u\eta_h)u_x h_x + (-\xi_u + u\eta_u)u_x^2 + \eta_h h_x^2, \\ \varphi^t &= \varphi_t - uh\varphi_v + (-u\varphi_u - \xi_t + uh\xi_v + u\eta_t - u^2h\eta_v)u_x + (-\varphi_u + \eta_t - uh\eta_v)h_x \\ &+ (\xi_u - 2u\eta_u)u_x h_x + \varphi_h h_t + (-\xi_h + u\eta_h)u_x h_t + (u\xi_u - u^2\eta_u)u_x^2 \\ &- \eta_u h_x^2 + \eta_h h_x h_t, \\ H^x &= H_x + hH_v + (H_h - \xi_x - h\xi_v)h_x + (-\eta_x - h\eta_v)h_t - \eta_h h_x h_t \\ &- \eta_u u_x h_t + H_u u_x - \xi_h h_x^2 - \xi_u u_x h_x, \\ G^x &= G_x + hG_v - h\xi_x - h^2\xi_v + uh\eta_x + uh^2\eta_v + (G_h - h\xi_h + uh\eta_h)h_x \\ &+ (G_u - h\xi_u + uh\eta_u)u_x, \\ G^t &= G_t - uhG_v + uh^2\xi_v + uh\eta_t - u^2h^2\eta_v - h\xi_t + (-uG_u + uh\xi_u - u^2h\eta_u)u_x \\ &+ (G_h - h\xi_h + uh\eta_h)h_t + (-G_u + h\xi_u - uh\eta_u)h_x. \end{aligned}$$

In order that the generator V leaves invariant the system (3), we need the symmetry conditions to be satisfied:

$$Pr^{(1)}V(E_j)/(E_i = 0, i = 1, 2, 3) = 0, \quad j = 1, 2, 3 \quad (6)$$

where

$$\begin{aligned} E_1 &= v_x - h, \\ E_2 &= v_t + uh, \\ E_3 &= u_t + uu_x + h_x. \end{aligned}$$

The symmetry conditions lead us to

$$\begin{cases} -H + G_{/E_i=0}^x &= 0, \\ h\varphi + uH + G_{/E_i=0}^t &= 0, \\ \varphi^t + H^x + u_x\varphi + u\varphi_{/E_i=0}^x &= 0, \quad i = 1, 2, 3. \end{cases} \tag{7}$$

Substituting φ^x, \dots, G^t by their expressions in the above system and equating the coefficients of monomials in the first partial derivatives of u and h , we get the following system of determining equations

$$\begin{aligned} \eta_u = \eta_h = \xi_h = \xi_u = G_u = G_h &= 0, \\ -H + G_x + hG_v - h\xi_x - h^2\xi_v + uh\eta_x + uh^2\eta_v &= 0, \\ h\varphi + uH + G_t - uhG_v + uh^2\xi_v + uh\eta_t - h\xi_t - u^2h^2\eta_v &= 0, \\ \varphi_t + H_x + hH_v + u\varphi_x &= 0, \\ -\xi_t + u\eta_t + H_u + \varphi - u\xi_x + u^2\eta_x &= 0, \\ -\varphi_u + \eta_t + H_h - \xi_x - h\xi_v + u\varphi_h + u\eta_x &= 0, \\ \varphi_h - \eta_x - h\eta_v &= 0. \end{aligned}$$

Consequently, we get a six parameter (a_1, \dots, a_6) group admitted by the system (3) with infinitesimals

$$\begin{aligned} \xi &= a_4t + a_3x + a_1, \\ \eta &= a_5t + a_2, \\ \varphi &= (a_3 - a_5)u + a_4, \\ H &= 2(a_3 - a_5)h, \\ G &= (3a_3 - 2a_5v + a_6, \end{aligned}$$

where a_1, \dots, a_6 are arbitrary constants. Hence, the Lie symmetry algebra of the auxiliary system (3) is spanned by the six generators

$$\begin{aligned} V_1 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ V_2 &= x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + 2h \frac{\partial}{\partial h} + 3v \frac{\partial}{\partial v}, \\ V_3 &= \frac{\partial}{\partial x}, \\ V_4 &= t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} - 2h \frac{\partial}{\partial h} - 2v \frac{\partial}{\partial v}, \\ V_5 &= t \frac{\partial}{\partial t}, \\ V_6 &= \frac{\partial}{\partial v}. \end{aligned}$$

Because the infinitesimals ξ, η, φ and h do not depend explicitly on the potential v , i.e.,

$$\left(\frac{\partial \xi}{\partial v}\right)^2 + \left(\frac{\partial \eta}{\partial v}\right)^2 + \left(\frac{\partial \varphi}{\partial v}\right)^2 = 0,$$

we see that $V_i, i = 1, \dots, 6$, define only point symmetries admitted by the shallow water equations with generators

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial x}, \\ Y_2 &= t \frac{\partial}{\partial t}, \\ Y_3 &= x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + 2h \frac{\partial}{\partial h}, \\ Y_4 &= t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \\ Y_5 &= t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u} - 2h \frac{\partial}{\partial h}. \end{aligned}$$

Commutation relations between these vector fields are given by the following table. The entry in row i and column j representing $[Y_i, Y_j]$:

	Y_1	Y_2	Y_3	Y_4	Y_5
Y_1	0	0	Y_1	0	0
Y_2	0	0	0	Y_1	Y_2
Y_3	$-Y_1$	0	0	$-Y_4$	0
Y_4	0	$-Y_1$	Y_4	0	$-Y_4$
Y_5	0	$-Y_2$	0	Y_4	0

Consequently, the symmetry algebra obtained in [4] by direct construction of Lie point symmetry admitted by shallow water equations is a subalgebra of the algebra spanned by $Y_i, i = 1, \dots, 5$, constructed by passing through the potential system (3).

3 Symmetry Reduction

It is clear that the shallow water equations do not admit potential symmetries, that is, the infinitesimals ξ, η and φ do not depend explicitly on the potential v . From the vector fields obtained in the previous section, we conclude that all point symmetries admitted by the shallow water equations are of projectable type in which the action on the independent variables do not depend on the dependent variables.

3.1 Reduction with Y_1

The similarity variables are:

$$y = t, \quad Z = u, \quad \text{and} \quad W = h.$$

Substituting similarity variables in system (1) and using chain rule imply that u and h must be constants. Thus a trivial constant solutions are obtained.

3.2 Reduction with Y_2

The similarity variables are:

$$y = x, \quad Z = u, \quad \text{and} \quad W = h.$$

Similarly to the previous case we obtain the constant solutions u and h .

3.3 Reduction with Y_3

As the symmetry is

$$x \frac{\partial}{\partial x} + u \frac{\partial}{\partial u} + 2h \frac{\partial}{\partial h},$$

the similarity variables are precisely

$$X = t, \quad Y = x^{-1}u, \quad \text{and} \quad W = x^{-1}h^{\frac{1}{2}}.$$

Then by chain rule, we get

$$\begin{aligned} u &= xY, & u_t &= xY', & u_x &= Y, \\ h &= x^2W^2, & h_t &= 2x^2WW', & h_x &= 2xW^2, \end{aligned}$$

where the prime denotes the derivative with respect to X . Substituting in (1), we obtain the reduced system

$$\begin{cases} 2W' + 3YW & = 0, \\ Y' + Y^2 + 2W^2 & = 0. \end{cases}$$

Since the first equation implies $Y = -\frac{2W'}{3W}$, from the second equation we get

$$W'' = \frac{5}{3} \frac{(W')^2}{W} + 3W^3.$$

If we put $W = \varphi^{-1}$, then φ satisfies the ordinary differential equation

$$\varphi'' = -3W^3(\varphi')^3 - \frac{5}{3}W\varphi'.$$

The general solution of this equation is

$$\varphi(W) = -3 \int \frac{v^2}{\sqrt{9 + cv^2}} dv, \quad v = W^{-\frac{1}{3}}.$$

Case.1: $c = 0$. In this case $\varphi = -\frac{1}{3W}$. Since $X = \varphi(W)$ thus $W = -\frac{1}{3X}$. Consequently, the exact solution of the shallow water equations obtained in this case is:

$$u(x, t) = \frac{2x}{3t}, \quad \text{and} \quad h(x, t) = \left(\frac{x}{3t}\right)^2.$$

Case.2: $c \neq 0$. In this case

$$\begin{cases} \varphi_c(W) &= -\frac{9v}{2c}\sqrt{1 + \frac{c}{9}v^2} + \frac{81}{2c\sqrt{c}} \arctan\left(\frac{\sqrt{c}}{3}v\right), \\ v &= W^{-\frac{1}{3}}. \end{cases}$$

The implicit solutions of the shallow water equations defined by the inverse function of φ_c cannot be found by the symmetry algebra obtained in [4]. Then it gives arise of a new exact solution.

3.4 Reduction with Y_4

The similarity variables are

$$X = t, \quad Y = h, \quad \text{and} \quad W = x - tu.$$

Chain rule implies

$$\begin{aligned} u_t &= -t^{-2}x + t^{-2}W - t^{-1}W', & u_x &= t^{-1}, \\ h_t &= Y', & h_x &= 0. \end{aligned}$$

Substituting in (1), we obtain

$$\begin{cases} tY' + Y &= 0, \\ W' &= 0. \end{cases}$$

The exact solutions constructed in this case are

$$u(x, t) = \frac{x - a}{t}, \quad \text{and} \quad h(x, t) = \frac{b}{t},$$

where a and b are arbitrary constants.

3.5 Reduction with Y_5

The symmetry Y_5 is given by

$$t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - 2h\frac{\partial}{\partial h},$$

the invariants are

$$X = x, \quad Y = tu, \quad \text{and} \quad W = t^2h.$$

Using chain rule, we get

$$\begin{aligned} u_t &= -t^{-2}Y, & u_x &= t^{-1}Y', \\ h_t &= -2t^{-3}W, & h_x &= t^{-2}W'. \end{aligned}$$

Substituting in (1), we obtain the reduced system :

$$\begin{cases} -W + (YW)' &= 0, \\ -Y + YY' + W' &= 0. \end{cases}$$

The general solutions of the above system are

$$\begin{cases} W &= \left(\frac{1}{3}X + d\right)^2, \\ Y &= \frac{1}{3}X + d, \end{cases}$$

where d is an arbitrary constant. Consequently, the exact solution of the shallow water equations obtained in this case is

$$u(x, t) = \frac{x + d}{3t}, \quad \text{and} \quad h(x, t) = \left(\frac{x + d}{3t}\right)^2.$$

4 Conclusion

In this work we concentrated on the one-dimensional shallow water equations. Using group analysis all similarity reductions of this system are found. Consequently, construction of some exact solutions was done. The search for Lie point symmetries of the shallow water equations was done in [4] but without using the system (3). The symmetry algebra obtained in [4] is a subalgebra which is obtained here. The method used above for determining new symmetries for shallow water equations arising from the first equation can also be extended to the second equation of (1). The difficulty is that the system of determining equations is more complicated than one obtained by considering the first equation. If we look only for projectable symmetries, the system of determining equations is completely solved and it does yield new symmetries.

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