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Rolle's Theorem For Complex Roots Of Polynomials Of Small Degree*

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Abstract

There are polynomials of odd degree d, for which Rolle's property does not hold, i.e., polynomials such that for some pair of roots their formal derivative has no root "in between" them. We describe in detail a counter-example for degree $d = 7$ and show that we cannot adapt the method to get a counter-example when $d = 5$. This case is the smallest unknown case. However, we prove that all polynomials of degree 5 with real coefficients do have Rolle's property.

1 Introduction

All polynomials in this paper have complex coefficients. Let $R(z)$ be a polynomial. Let a, b be two distinct complex numbers. We say that $R(z)$ has a root r "in between" a and b if $\pi(r)$ is in the segment $\{a + t(b - a) \,|\, 0 < t < 1\}$ where π is the orthogonal projection over the line determined by a and b. We say that R has Rolle's property when the first derivative R' of R has a root in between every pair of given (distinct) roots of R.

We know [1, Theorem 1, Theorem 2] that all polynomials of degree at most equal to 4, and that all polynomials of whatever degree but having at most three distinct roots have Rolle's property. The first fact comes essentially from a result [2, Problem 150, p.60 and p.238] reported by Polya. See also [1, Proposition 2.1] for some general sufficient conditions such that the property holds, and for a detailed bibliography on the subject.

For completeness we recall that it is easy to get infinite families of polynomials with Rolle's property. Consider, for example, the important case of degree 5. We have the following one-parameter $a \in \mathbb{C}$ family:

$$P = \frac{1}{5}x^5 + \frac{1}{4}(-3a - b)x^4 + \frac{1}{3}(3a^2 + 3ab)x^3 + \frac{1}{2}(-a^3 - 3a^2b)x^2 + a^3bx + K,$$

where K is chosen in order to have $P(i) + P(-i) = 0$ and $b = f(a)$ is chosen so that $P(i) - P(-i) = 0$. More precisely, we get $K = \frac{3}{4}a - \frac{1}{2}a^3 - \frac{3}{2}a^2b + \frac{1}{4}b$ and we get also

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$b = f(a) = \frac{5a^2-1}{5a(a^2-1)}$. Observe that $P^{'}(x) = (x-a)^3(x-b)$ is a polynomial with at most two distinct roots. Thus, the family has Rolle's property by [1, Theorem 2].

The idea under the known counter-examples comes from Denis Simon [3] (his private e-mail considers several concrete cases of polynomials P of degree d with $d \leq 30$ but does not contain a general proof). He informs me the following procedure which may lead to such counter-examples when the degree d of P is an odd (and "small") number. However, no proof exists to show that the procedure works for general odd numbers. In other words it is not known for a general odd number whether counter-examples do exist to the Rolle's property. We are just able to construct with many computer calculations some counterexamples for specific small odd numbers. Here below follows a sketch of the procedure and a further consideration of the special cases $d \in \{7, 5\}$.

(a) We take $P' = ((x-a)^2 + b^2)^k$ for some real numbers a, b to be determined, and for a given positive integer k.

(b) We force P, the integral of P', to be divisible by $x^2 + 1$ so that we may consider the roots of P' in between $-i$ and i.

(c) If possible we get real a, b from (b). This guarantees that we get a polynomial P with real coefficients whose derivative has only roots with imaginary part equal to b (or to $-b$). We compare $abs(b)$ with 1. If $abs(b) \geq 1$ then we have a counter-example.

More precisely, it suffices to study in (b) the relation $R(a, b) = 0$ that eliminates the imaginary part of $P(i)$. If we get such real numbers a, b we can always translate P such that (b) holds. In order to choose the maximal possible b we can check the $b's$ that are solutions of $D(b) = 0$, where D is the discriminant of $R(a, b)$ relative to a. So we have a finite number of possibilities to try.

This procedure gives already a counter-example for degree $d = 7$. This requires some symbolic and numeric computations in Maple. The computations get more complicated for higher odd values of d and it seems difficult to give a general result. Moreover, nothing is known when the degree d is an even number > 4. Furthermore, we do not know about a procedure to construct a possible counter-example, even in the special case when $d = 6$. However, for $d = 5$, the procedure fails since we get a polynomial P with real coefficients for which Rolle's property is true since the values of $abs(b)$ are < 1 and we may check, after some computations, that the property holds also for the other pairs of roots.

More precisely, we get $b^2 = \frac{3}{5}$ and $a^2 = \frac{2}{5}$ so that

$$P = \frac{1}{5}x^5 - \frac{1}{5}\sqrt{10}x^4 + \frac{6}{5}x^3 - \frac{2}{5}\sqrt{10}x^2 + x - \frac{1}{5}\sqrt{10},$$

that has the factorization

$$P = \frac{1}{5}(x^3 - \sqrt{10}x^2 + 5x - \sqrt{10})(x^2 + 1)$$

while P' factors as

$$P' = \frac{1}{25}(5x^2 - 2\sqrt{10}x + 5)^2.$$

The object of this paper is to display Simon's degree 7 counter-example and to prove the following new result in the unknown case of degree 5 :

THEOREM 1. Let P be a polynomial of degree 5 that has real coefficients. Then P has Rolle's property.

2 Simon's Counter-Example

We follow the steps described in the Introduction. The discriminant reads

$$d_7 = \frac{-4194304}{52521875}(-5+21b^2-35b^4+35b^6)(-400+1680b^2-2548b^4+1225b^6)^2.$$

Two real solutions b satisfy $b^2 > 1$ so that by choosing one of them we have the counter-example. More precisely, our b (approximately $b = 1.052910994$) satisfies $-400 + 1680b^2 - 2548b^4 + 1225b^6 = 0$. By choosing a real number a such that $P(i) = 0$ i.e., (approximately) $a = 1.051399647$, we obtain $K = -7.205400746$ and we get the counter-example:

$$P = \frac{1}{7}x^7 - ax^6 + (\frac{3}{5}b^2 + 3a^2)x^5 + (-3ab^2 - 5a^3)x^4 + (6a^2b^2 + b^4 + 5a^4)x^3 + (-3ab^4 - 3a^5 - 6a^3b^2)x^2 + (a^6 + 3a^4b^2 + 3a^2b^4 + b^6)x + K.$$

3 Main Condition for Rolle's Property

Let $P_1 = t^4 - s_1t^3 + s_2t^2 - s_3t + s_4$ be a monic polynomial of degree 4. We take a formal antiderivative

$$P = \frac{1}{5}t^5 - \frac{1}{4}s_1t^4 + \frac{1}{3}s_2t^3 - \frac{1}{2}s_3t^2 + s_4t - s_5$$

for some constant s_5. In order to prove that P has Rolle's property it suffices to take $P(i) = 0$ and $P(-i) = 0$ and to prove that at least one root of $P_1 = P^{'}$ have the absolute value of the imaginary part less than 1. Observe that under the above condition

$$P(i) - P(-i) = 0 \tag{1}$$

trivially holds. So from (1) we get after some computation the condition

$$\delta := 15s_4 - 5s_2 + 3 = 0. \tag{2}$$

In other words, any polynomial P such that $P(i) = 0 = P(-i)$ also satisfies (2). Moreover, any polynomial P such that $P(i) = 0 = P(-i)$ and such that at least one of the roots of $P^{'}$ has the absolute value of the imaginary part less than 1 satisfies Rolle's property.

Assume that P is a polynomial of degree 5 such that $P(i) = 0 = P(-i)$. For example assume that P is a polynomial of degree 5 with real coefficients such that $P(i) = 0$. But assume also, to the contrary, that the absolute values of the imaginary parts of all the roots of $P^{'}$ are at least equal to 1. If we are able to prove that $\delta \neq 0$ (e.g., if we prove that $\delta > 0$) then we get Rolle's property for P.

4 Proof of Theorem

Following the criterion in the preceding section we proceed to assume that $P(i) = 0$ so that $P(i) = 0 = P(-i)$ since P has real coefficients. We assume that for some real numbers x, y, z, w the roots of P' are $z_1 = x + yi, z_2 = z + wi$ and their complex conjugates $z_3 = x - yi$ and $z_4 = z - wi$. So our main condition that the roots have the absolute values of their imaginary part not less than 1 becomes

$$y^2 \geq 1 \text{ and } w^2 \geq 1.$$

On the other hand, after some computation we get

$$\delta = 15(x^2z^2 + x^2w^2 + y^2z^2 + y^2w^2) - 5(x^2 + y^2 + z^2 + w^2) - 20xz + 3. \tag{3}$$

We have then the following inequalities: $15x^2w^2 \geq 15x^2$ since $w^2 \geq 1$ and symmetrically $15y^2z^2 \geq 15z^2$ since $y^2 \geq 1$ so that

$$(15x^2w^2 - 5x^2) + (15y^2z^2 - 5z^2) \geq 10x^2 + 10z^2. \tag{4}$$

Observe that $10x^2 + 10z^2 - 20xz = 10(x - z)^2$ so that from (3) and (4) we get

$$\delta \geq 10(x - z)^2 + 15x^2z^2 + 5y^2(w^2 - 1) + 5w^2(y^2 - 1) + 8, \tag{5}$$

since $5y^2w^2 \geq 5$. It follows then immediately that

$$\delta \geq 8 > 0.$$

This finishes the proof of the theorem.

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