

Existence And Uniqueness Of Mild Solutions Of Second Order Volterra Integrodifferential Equations With Nonlocal Conditions*

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Abstract

In this paper, we study the existence and uniqueness of mild solutions for second order initial value problems, with nonlocal conditions, by using the Banach fixed point theorem and the theory of strongly continuous cosine family.

1 Introduction

Let X be a Banach space with norm $\|\cdot\|$ and throughout this paper we assume the notation $J = [0, b]$. Let $B = C(J, X)$ be Banach space of all continuous functions from J into X , endowed with the norm

$$\|x\|_b = \sup\{\|x(t)\| : x \in B, \quad t \in J\}.$$

In the present paper we consider the following second order nonlinear integrodifferential equations with nonlocal conditions:

$$x''(t) = Ax(t) + f(t, x(t), \int_0^t k(t, s, x(s))ds), \quad t \in J, \quad (1)$$

$$x(0) = x_0 + q(x), \quad (2)$$

$$x'(0) = y_0 + p(x), \quad (3)$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space X , $f : J \times X \times X \rightarrow X$, $k : J \times J \times X \rightarrow X$, $q, p : B \rightarrow X$ are appropriate continuous functions, and x_0, y_0 are given elements of X .

The work on nonlocal initial value problem (IVP) was initiated by Byszewski. In [3], Byszewski, using the method of semigroups and the Banach fixed point theorem proved the existence and uniqueness of mild, strong and classical solution of first order

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IVP. For the importance of nonlocal conditions in different fields, the interesting reader is referred to [1,2,4,8] and the references cited therein.

The aim of this paper is to study the global existence of solutions of the equations (1)–(3). The main tool used in our analysis is based on the contraction mapping principle and the cosine function theory. The theorem proved in this paper generalizes some results obtained by Hernandez in [5]. We are motivated by the work of Hernandez in [5] and influenced by the work of Byszewski [3].

The paper is organized as follows: In Section 2, we present the preliminaries and hypotheses. Section 3 deals with the main results. Finally, in Section 4, we give an example of ‘wave’ equation to illustrate the application of our theorem.

2 Preliminaries and Hypotheses

In many cases it is advantageous to treat second order abstract differential equations directly rather than to convert it into order systems. A useful technique for the study of abstract second order equations is the theory of strongly continuous cosine family. We refer the reader to [6,10,11] for the necessary concepts about cosine functions. Next, we only mention a few results and notations needed to establish our results. A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators mapping the Banach space X into itself is called a strongly continuous cosine family if and only if

- (a) $C(0) = I$ (I is the identity operator);
- (b) $C(t)x$ is strongly continuous in t on \mathbb{R} for each fixed $x \in X$;
- (c) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

If $\{C(t) : t \in \mathbb{R}\}$ is a strongly continuous cosine family in X , then $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family, is defined by

$$S(t)x = \int_0^t C(s)x ds, \quad x \in X, \quad t \in \mathbb{R}. \quad (4)$$

For a closed operator $G : D(G) \subset X \rightarrow X$ we denote by $[D(G)]$ the space $D(G)$ endowed with the graph norm $\|\cdot\|_G$. The infinitesimal generator $A : X \rightarrow X$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$Ax = \frac{d^2}{dt^2}C(t)x|_{t=0}, \quad x \in D(A),$$

where $D(A) = \{x \in X : C(\cdot)x \in C^2(\mathbb{R}, X)\}$. Moreover, let M and N be two positive constants with $M \geq 1$, such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$ for every $t \in J$.

In this paper, $[D(A)]$ is the space $D(A) = \{x \in X : C(\cdot)x \in C^2(\mathbb{R}, X)\}$, endowed with the norm

$$\|x\|_A = \|x\| + \|Ax\|, \quad x \in D(A).$$

We shall also make use of the space $E = \{x \in X : C(\cdot)x \in C^1(\mathbb{R}, X)\}$. We refer to Kisinsky [7] for details, that E endowed with the norm

$$\|x\|_E = \|x\| + \sup_{t \in J} \|AS(t)x\|, \quad x \in E,$$

is a Banach space. The operator-valued function $g(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$ is a strongly continuous group of bounded linear operators on the space $E \times X$ generated by the operator $\mathcal{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$ defined on $D(A) \times E$. From this, it follows that $AS(t) : E \rightarrow X$ is a bounded operator and that $AS(t)x \rightarrow 0$, as $t \rightarrow 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \rightarrow X$ is a locally integrable, then

$$y(t) = \int_0^t S(t-s)x(s)ds$$

defines an E -valued continuous function, which is a consequence of the fact that

$$\int_0^t g(t-s) \begin{bmatrix} 0 \\ x(s) \end{bmatrix} ds = \begin{bmatrix} \int_0^t S(t-s)x(s) ds \\ \int_0^t C(t-s)x(s) ds \end{bmatrix}$$

defines an $E \times X$ -valued continuous function.

DEFINITION. A function $x \in B$ is a mild solution of the problem (1)–(3) if condition (2) is verified and satisfy

$$\begin{aligned} x(t) &= C(t)[x_0 + q(x)] + S(t)[y_0 + p(x)] \\ &+ \int_0^t S(t-s)f\left(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau\right)ds, \quad t \in J. \end{aligned} \quad (5)$$

We list the following hypotheses for our convenience.

(H₁) X is a Banach space with norm $\|\cdot\|$ and $x_0, y_0 \in X$.

(H₂) $t \in J$ and $B_r = \{z : \|z\| \leq r\} \subset X$.

(H₃) $f : J \times X \times X \rightarrow X$ is continuous in t on J and there exist positive constants L_f^1, L_f^2 such that

$$\|f(t, x_1, y_1) - f(t, x_2, y_2)\| \leq L_f^1 \|x_1 - x_2\| + L_f^2 \|y_1 - y_2\|.$$

(H₄) $k : J \times J \times X \rightarrow X$ is continuous in t, s on J and there exists a constant $K > 0$ such that

$$\|k(t, s, x_1) - k(t, s, x_2)\| \leq K \|x_1 - x_2\|.$$

(H₅) $q, p : B \rightarrow X$ are continuous and there exist constants $L_q, L_p > 0$ such that

$$\begin{aligned} \|q(x_1) - q(x_2)\| &\leq L_q \|x_1 - x_2\|_b \quad \text{and} \\ \|p(x_1) - p(x_2)\| &\leq L_p \|x_1 - x_2\|_b, \quad \text{for } x_1, x_2 \in C(J, B_r). \end{aligned}$$

(H₆) A is the infinitesimal generator of strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ and $S(t)$, the sine function associated with $C(t)$, which is defined in (4).

(H₇)

$$\begin{aligned}
L_1 &= \max_{t \in J} \|f(t, 0, 0)\|, \\
K_1 &= \max_{t \in J} \|k(t, s, 0)\|, \\
Q &= \max_{x \in C(J, B_r)} \|q(x)\|, \\
P &= \max_{x \in C(J, B_r)} \|p(x)\|.
\end{aligned}$$

(H₈) The constants $\|x_0\|$, $\|y_0\|$, r , b , L_f^1 , L_f^2 , K , L_q , L_p , M , N , L_1 , K_1 , Q , and P satisfy the following inequalities:

$$\begin{aligned}
M[\|x_0\| + Q] + N[\|y_0\| + P] + N[L_f^1 r b + L_f^2 K r b^2 + L_f^2 K_1 b^2 + L_1 b] &\leq r, \\
[ML_q + N(L_p + L_f^1 b + L_f^2 K b^2)] &< 1.
\end{aligned}$$

We need the following integral inequality, often referred to as Gronwall-Bellman inequality [9, p.11].

LEMMA 1. Let u and f be continuous functions defined on \mathbb{R}_+ and c be a nonnegative constant. If

$$u(t) \leq c + \int_0^t f(s)u(s)ds,$$

for $t \in \mathbb{R}_+$, then

$$u(t) \leq c \exp\left(\int_0^t f(s)ds\right),$$

for $t \in \mathbb{R}_+$.

3 Existence of Mild Solution

Now we establish our first result.

THEOREM 1. Suppose that the hypotheses [H₁] – [H₈] hold, then the problem (1)–(3) has a unique mild solution on J .

PROOF. Let $Z = \{z \in B | z \in B_r\}$ and define an operator $F : Z \rightarrow Z$ by

$$\begin{aligned}
(Fz)(t) &= C(t)[x_0 + q(z)] + S(t)[y_0 + p(z)] \\
&\quad + \int_0^t S(t-s)f\left(s, z(s), \int_0^s k(s, \tau, z(\tau))d\tau\right)ds, \quad t \in J. \quad (6)
\end{aligned}$$

We shall first show that F maps Z into itself. To this end, from the definition of the

operator F in (6) and our hypotheses, we obtain

$$\begin{aligned}
 \|(Fz)(t)\| &\leq \|C(t)[x_0 + q(z)] + S(t)[y_0 + p(z)]\| \\
 &\quad + \int_0^t \|S(t-s)f\left(s, z(s), \int_0^s k(s, \tau, z(\tau))d\tau\right)\| ds \\
 &\leq M[\|x_0\| + Q] + N[\|y_0\| + P] \\
 &\quad + N \int_0^t \left[\|f\left(s, z(s), \int_0^s k(s, \tau, z(\tau))d\tau\right) - f(s, 0, 0) + f(s, 0, 0)\| \right] ds \\
 &\leq M[\|x_0\| + Q] + N[\|y_0\| + P] \\
 &\quad + N \int_0^t \left[L_f^1 r + L_f^2 K r b + L_f^2 K_1 b + L_1 \right] ds \\
 &\leq M[\|x_0\| + Q] + N[\|y_0\| + P] \\
 &\quad + N \left[L_f^1 r b + L_f^2 K r b^2 + L_f^2 K_1 b^2 + L_1 b \right] \\
 &\leq r,
 \end{aligned} \tag{7}$$

for $z \in Z$ and $t \in J$. Hence $\|Fz\|_b \leq r$. Therefore, the equation (7) shows that the operator F maps Z into itself.

Now, we shall show that F is a contraction on Z . Then for this every $z_1, z_2 \in Z$ and $t \in J$, we have

$$\begin{aligned}
 \|(Fz_1)(t) - (Fz_2)(t)\| &\leq \|C(t)\| \|q(z_1) - q(z_2)\| + \|S(t)\| \|p(z_1) - p(z_2)\| \\
 &\quad + \int_0^t \|S(t-s) \left[f\left(s, z_1(s), \int_0^s k(s, \tau, z_1(\tau))d\tau\right) \right. \\
 &\quad \left. - f\left(s, z_2(s), \int_0^s k(s, \tau, z_2(\tau))d\tau\right) \right]\| ds \\
 &\leq [ML_q + NL_p] \|z_1 - z_2\|_b \\
 &\quad + N \int_0^t \left[L_f^1 \|z_1(s) - z_2(s)\| + L_f^2 \int_0^s K \|z_1(\tau) - z_2(\tau)\| d\tau \right] ds \\
 &\leq [ML_q + N(L_p + L_f^1 b + L_f^2 K b^2)] \|z_1 - z_2\|_b.
 \end{aligned} \tag{8}$$

If we take $l = [ML_q + N(L_p + L_f^1 b + L_f^2 K b^2)]$, then

$$\|Fz_1 - Fz_2\|_b \leq l \|z_1 - z_2\|_b$$

with $0 < l < 1$. This shows that the operator F is a contraction on the complete metric space Z . By the Banach fixed point theorem, the function F has a unique fixed point in the space Z and this point is the mild solution of problem (1)–(3) on J .

COROLLARY 1. Assume that the hypotheses $[H_1] - [H_8]$ hold. Furthermore, if

(i) $f : J \times X \times X \rightarrow X$ is continuous and there exist constant $L_t > 0$ such that

$$\|f(t_1, x_1, y_1) - f(t_2, x_1, y_1)\| \leq L_t |t_1 - t_2|,$$

(ii) $k : J \times J \times X \rightarrow X$ is continuous and there exist constant $K_t > 0$ such that

$$\|k(t_1, s, x_1) - k(t_2, s, x_1)\| \leq K_t |t_1 - t_2|,$$

(iii) $(y_0 + p(x), x_0 + q(x)) \in D(A) \times E$.

Then $x(\cdot)$ is Lipschitz continuous on J .

PROOF. Since all the assumptions of Theorem 1 are satisfied then the problem (1)–(3) has a unique mild solution belonging to Z which we denote it by x . Now we will show that $x(\cdot)$ is Lipschitz continuous on J . Take

$$L_2 = \max_{t \in J} \|f(t, x(t), 0)\|,$$

$$K_2 = \max_{t \in J} \|k(t, s, x(s))\|.$$

Let $t \in J$ and θ is a real number such that $t + \theta \in J$. Then, for $t \in J$, and using the strong continuity of the cosine family, we obtain

$$\begin{aligned} \|x(t + \theta) - x(t)\| &\leq C_1 \theta + \int_0^\theta \|S(t + \theta - s)\| \\ &\quad \times \left[\|f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau) - f(s, x(s), 0)\| + \|f(s, x(s), 0)\| \right] ds \\ &\quad + \int_0^t \|S(t - s)\| \left[\|f(s + \theta, x(s + \theta), \int_0^{s+\theta} k(s + \theta, \tau, x(\tau))d\tau) \right. \\ &\quad \left. - f(s, x(s), \int_0^s k(s, \tau, x(\tau))d\tau) \right] ds \\ &\leq C_1 \theta + \int_0^\theta N \left[L_t |s - s| + L_f^1 \|x(s) - x(s)\| \right. \\ &\quad \left. + L_f^2 \left\| \int_0^s k(s, \tau, x(\tau))d\tau - 0 \right\| + L_2 \right] ds \\ &\quad + \int_0^t N \left[L_t |s + \theta - s| + L_f^1 \|x(s + \theta) - x(s)\| \right. \\ &\quad \left. + L_f^2 \left\| \int_0^s k(s + \theta, \tau, x(\tau))d\tau - \int_0^s k(s, \tau, x(\tau))d\tau \right\| \right. \\ &\quad \left. + L_f^2 \left\| \int_s^{s+\theta} k(s + \theta, \tau, x(\tau))d\tau \right\| \right] ds \\ &\leq C_2 \theta + N L_f^1 \int_0^t \|x(s + \theta) - x(s)\| ds, \end{aligned} \tag{9}$$

where C_1, C_2 are constants independents of θ and $t \in J$. Using Lemma 1 (with $c = C_2 \theta$), we get

$$\|x(t + \theta) - x(t)\|_b \leq C_2 \theta e^{N L_f^1 b}, \quad \text{for } t \in J.$$

Therefore, $x(\cdot)$ is Lipschitz continuous on J .

4 Application

In this section, we give an example of ‘wave’ equation to illustrate of our main theorem. Consider the following partial integrodifferential equation:

$$\frac{\partial^2 w(t, u)}{\partial t^2} = \frac{\partial^2 w(t, u)}{\partial u^2} + \mu\left(t, w(t, u), \int_0^t a(t, s, w(s, u)) ds\right),$$

$$t \in J, \quad u \in I = [0, \pi], \quad (10)$$

$$w(t, 0) = w(t, \pi) = 0, \quad t \in J, \quad (11)$$

$$w(0, u) = x_0(u) + \sum_{i=1}^n \alpha_i w(t_i, u), \quad u \in I, \quad (12)$$

$$\frac{\partial w(t, u)}{\partial t} \Big|_{t=0} = y_0(u) + \sum_{i=1}^k \beta_i w(s_i, u), \quad u \in I, \quad (13)$$

where $\mu : J \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $a : J \times J \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous and $0 < t_i, s_i < b$, $\alpha_i, \beta_i \in \mathbb{R}$ are prefixed numbers. Let us take $X = L^2([0, \pi])$. We define the operator $A : D(A) \subset X \rightarrow X$ by $Aw = w_{uu}$, where $D(A) = \{w(\cdot) \in X : w(0) = w(\pi) = 0\}$. It is well known that A is the generator of strongly continuous cosine function $\{C(t) : t \in \mathbb{R}\}$ on X . Furthermore, A has discrete spectrum, the eigenvalues are $-n^2, n \in \mathbb{N}$, with corresponding normalized characteristics vectors $w_n(u) := \sqrt{\frac{2}{\pi}} \sin(nu)$, $n = 1, 2, 3, \dots$, and the following conditions hold :

- (i) $\{w_n : n \in \mathbb{N}\}$ is an orthonormal basis of X .
- (ii) If $w \in D(A)$ then $Aw = -\sum_{n=1}^{\infty} n^2 \langle w, w_n \rangle w_n$.
- (iii) For $w \in X$, $C(t)w = \sum_{n=1}^{\infty} \cos(nt) \langle w, w_n \rangle w_n$. Moreover, from these expression, it follows that $S(t)w = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle w, w_n \rangle w_n$, that $S(t)$ is compact for every $t > 0$ and that $\|C(t)\| \leq 1$ and $\|S(t)\| \leq 1$ for every $t \in J$.
- (iv) If H denotes the group of translations on X defined by $H(t)x(u) = \tilde{x}(u + t)$, where \tilde{x} is the extension of x with period 2π , then $C(t) = \frac{1}{2}(H(t) + H(-t))$. Hence it follows, see [6], that $A = G^2$, where G is the infinitesimal generator of the group H and that $E = \{x \in L^1(0, \pi) : x(0) = x(\pi) = 0\}$.

Define the functions $f : J \times X \times X \rightarrow X$, $a : J \times J \times X \rightarrow X$, and $q, p : C(J, X) \rightarrow X$ as follows

$$f(t, x, y)u = \mu(t, x(u), y(u)),$$

$$k(t, s, x)u = a(t, s, x(u)),$$

$$q(x)u = \sum_{i=1}^n \alpha_i x(t_i, u),$$

$$p(x)u = \sum_{i=1}^k \beta_i x(s_i, u), \quad u \in C(I : X),$$

for $0 < t_i, s_i < b$ and $0 \leq u \leq \pi$. Assume these functions satisfy the requirement of hypotheses. From the above choices of the functions and generator A , the equations (10)–(13) can be formulated as an abstract nonlinear second order integrodifferential equations (1)–(3) in Banach space X . Since all hypotheses of the Theorem [1] are satisfied, therefore, the Theorem 1 can be applied to guarantee the solution of the nonlinear partial integrodifferential equation (10)–(13).

References

- [1] K. Balachandran, Existence and uniqueness of mild and strong solutions of nonlinear integrodifferential equations with nonlocal condition, *Differential Equations and Dynamical Systems*, Vol. 6, 1/2(1998), 159–165.
- [2] M. Benchohra and S. K. Ntouyas, Existence of mild solutions of second order initial value problems for differential inclusions with nonlocal conditions. *Atti Sem. Mat. Fis. Univ. Modena* 49. 2(2001), 351–361.
- [3] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem. *J. Math. Anal. Appl.* 162, 2(1991), 494-505 .
- [4] L. Byszewski and V. Lakshmikantham, Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space. *Appl. Anal.*, 40, 1(1991), 11–19.
- [5] M. E. Hernandez, Existence of solutions to a second order partial differential equation with nonlocal conditions, *Electronic Journal of Differential Equations*, Vol. 2003(2003), No. 51, 1–10.
- [6] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Mathematics Studies, Vol. 108, North-Holland, Amsterdam, 1985.
- [7] J. Kiszyński, On second order Cauchy's problem in a Banach space, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronm. Phys.*, 18(1970) 371–374.
- [8] S. K. Ntouyas, Global existence results for certain second order delay integrodifferential equations with nonlocal conditions, *Dynam. Systems Appl.*, 7, 3(1998), 415–425.
- [9] B. G. Pachpatte; *Inequalities for Differential and Integral Equations*, Academic Press, New York, 1998.
- [10] C. C. Travis and G. F. Webb, Compactness, regularity, and uniform continuity properties of strongly continuous cosine families, *Houston J. Math.* 3(4) (1977), 555–567.
- [11] C. C. Travis and G. F. Webb, Cosine families and abstract nonlinear second order differential equations, *Acta Math. Acad. Sci. Hungaricae*, 32(1978), 76–96.