**Applied Mathematics E-Notes, 9(2009), 266-273 ©** ISSN 1607-2510
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# Strong Convergence Theorems Of Cesàro Mean Iterations Of Nonexpansive Mappings*

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Received 15 October 2008

**Abstract**

In a reflexive and strictly convex Banach space which has uniformly Gâteaux differentiable norm, we consider the problem of the convergence of the Cesàro mean iterations for non-expansive mappings. Under suitable conditions, it was proved that the iterative sequence converges strongly to a fixed point. The results presented in this paper also extend and improve some recent results.

## 1 Introduction

Let $X$ be a real Banach space and $T$ a mapping with domain $D(T)$ and range $R(T)$ in $X$. $T$ is called non-expansive if for any $x, y \in D(T)$ such that

$$\|Tx - Ty\| \leq \|x - y\|.$$

In 2000, Moudafi [5] introduced viscosity approximation methods and proved that if $X$ is a real Hilbert space, for given $x_0 \in C$, the sequence $\{x_n\}$ generated by the iteration process

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \quad n \geq 0,$$

where $f : C \to C$ is a contraction mapping and $\{\alpha_n\} \subseteq (0, 1)$ satisfies certain conditions, converges strongly to a fixed point of $T$ in $C$.

In last decades, many mathematical workers studied the iterative algorithms for various mappings, and obtained a series of good results, see [1, 6, 11].

In 2002, Xu [13] obtained the strong convergence of the iteration sequence $\{x_n\}$ given as follows:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} S^j x_n \quad \text{for } n = 0, 1, 2, \dots ,$$

*Mathematics Subject Classifications: 47H09, 47H10.

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for a non-expansive mapping in a uniformly convex and uniformly smooth Banach space. In 2004, Matsushita and Kuroiwa [3] extended the result of Xu to a non-expansive nonself-mapping in the same space. In 2007, Song and Chen [10] proposed the following viscosity iterative process $\{x_n\}$ given by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \quad n \geq 0,$$

and proved that the explicit process $\{x_n\}$ converges to a fixed point $p$ of $T$ in a uniformly convex Banach space with weakly sequentially continuous duality mapping and $\{\alpha_n\}$ satisfies certain conditions. Very recently, Wangkeeree [12] extended Song and Chen's result to non-expansive nonself-mapping. The author [12] also extended the result of Matsushita and Kuroiwa [3] to that of a Banach space. The purpose of this paper is to study the strong convergence for the iterative sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n \frac{1}{n+1}\sum_{j=0}^{n} T^j x_n. \tag{1}$$

Our results extend and improve the corresponding ones by [8, 9, 10, 13].

## 2 Preliminaries

Let $X$ be a real Banach space, and let $J$ denote the normalized duality mapping from $X$ into $2^{X^*}$ given by

$$J(x) = \{f \in X^* : \langle x, f\rangle = \|x\|\|f\|, \|f\| = \|x\|\}, \qquad \forall x \in X \tag{2}$$

where $X^*$ denotes the dual space of $X$ and $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. In the sequel, we shall denote the single-valued duality mapping by $j$, and denote $F(T) = \{x \in X : Tx = x\}$. When $\{x_n\}$ is a sequence in $X$, then $x_n \to x$ ($x_n \rightharpoonup x, x_n \rightharpoondown x$) will denote strong (weak, weak star)convergence of the sequence$\{x_n\}$ to $x$.

Recall that the norm is said to be uniformly Gâteaux differentiable if for each $x \in S_X := \{x \in X : \|x\| = 1\}$ the limit $\lim_{t\to 0} \frac{\|x+ty\|-\|x\|}{t}$ exists uniformly for $x \in S_X$. It is well known that every uniformly smooth Banach space has uniformly Gâteaux differentiable norm, and this implies that the duality mapping $J : X \to 2^{X^*}$ defined by (2) is single-valued and uniformly continuous on bounded subset of $X$ from the strong topology of $X$ to the weak star topology of $X^*$ (see e.g., [7]). A Banach space $X$ is said to be strictly convex if $\|x\| = \|y\| = 1$, $x \neq y$ implies $\|\frac{x+y}{2}\| < 1$. Now, let us first recall the following lemmas.

LEMMA 1.[4] Let $X$ be a real Banach space. For each $x, y \in X$, the following conclusion holds:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

LEMMA 2. [2] Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be three nonnegative real sequences satisfying $a_{n+1} \leq (1-t_n)a_n + b_n + c_n$ with $\{t_n\} \subset [0,1]$, $\sum_{n=0}^{\infty} t_n = \infty$, $b_n = o(t_n)$, and $\sum_{n=0}^{\infty} c_n < \infty$. Then $a_n \to 0$.

LEMMA 3. [11] Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\gamma_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$. Suppose that $x_{n+1} = \gamma_n x_n + (1-\gamma_n) y_n$ for all $n \in \mathbb{N}$ and

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

# 3 Main Results

Let $X$ be a Banach space, $C$ a nonempty closed convex subset of $X$, and $T : C \to C$ a non-expansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with contraction constant $\alpha$. For $t_n \in (0,1)$, define a mapping $T^f_{t_n} : C \to C$ by

$$T^f_{t_n}(z) = t_n f(z) + (1-t_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i z.$$

Clearly, for each $z \in C$ we have $T^f_{t_n}$ is a contractive mapping with contraction constant $t = 1 - t_n(1-\alpha)$. Hence, it follows from Banach's contractive principle that $T^f_{t_n}$ has a unique fixed point (say) $z_n \in C$, that is,

$$z_n = t_n f(z_n) + (1-t_n)\frac{1}{n+1}\sum_{i=0}^{n} T^i z_n. \tag{3}$$

Now if we set $T_n := \frac{1}{n+1}\sum_{i=0}^{n} T^i$, then the mapping $T : C \to C$ is said to satisfy *property (A)* if $C$ is bounded and for each $x \in C$, $\lim_{n\to\infty} \|T_n x - T(T_n x)\| = 0$.

There exist non-expansive mappings satisfying property (A).

EXAMPLE 4. Take $C = [0,1]$ and the norm is the ordinary Euclidean distance on the line. For each $x \in C$, $Tx = \frac{x}{2}$, then $T_n x = \frac{1}{n+1}\sum_{i=0}^{n} \frac{x}{2^i}$ and $T(T_n x) = \frac{1}{n+1}\sum_{i=0}^{n} \frac{x}{2^{i+1}}$. Hence we have

$$\begin{aligned}
\|T_n x - T(T_n x)\| &= \|\frac{1}{n+1}\sum_{i=0}^{n}\frac{x}{2^i} - \frac{1}{n+1}\sum_{i=0}^{n}\frac{x}{2^{i+1}}\| \\
&\leq \frac{1}{n+1}\sum_{i=0}^{n}\|\frac{x}{2^i} - \frac{x}{2^{i+1}}\| \\
&\leq \frac{1}{n+1}\left(1 - \left(\frac{1}{2}\right)^{n+1}\right) \to 0
\end{aligned}$$

as $n \to \infty$. That is $T$ satisfies property (A).

LEMMA 5. Let $C$ be a nonempty bounded closed convex subset of Banach space $X$ and $T : C \to C$ be a non-expansive mapping. For each $x \in C$ and the Cesàro means $T_n x = \frac{1}{n+1}\sum_{i=0}^{n} T^i x$,

(i) $T_n$ is non-expansive from $C$ to itself.

(ii) $\lim_{n\to\infty} \|T_{n+1}x - T_n x\| = 0$.

PROOF. (i) It is easy to see that $T_n$ is a mapping from $C$ to itself. We now prove that $T_n$ is non-expansive. In fact, since $T$ is non-expansive, then for each $x, y \in C$ we have

$$\|T_n x - T_n y\| \le \frac{1}{n+1}\sum_{i=0}^{n} \|T^i x - T^i y\| \le \frac{1}{n+1}\sum_{i=0}^{n} \|x - y\| = \|x - y\|. \tag{4}$$

This implies that $T_n$ is non-expansive.

(ii) Since $C$ is bounded, it is easy to prove that the sequence $\{T^i x\}$ is bounded and there exists a constant $M > 0$ such that $M > \max\{\sup_{x\in C} \|T^i x\| : i = 0, 1, ...\}$. Thus,

$$\begin{aligned}
\|T_{n+1}x - T_n x\| &= \|\frac{1}{n+2}\sum_{i=0}^{n+1} T^i x - \frac{1}{n+1}\sum_{i=0}^{n} T^i x\| \\
&\le \frac{\|\sum_{i=0}^{n+1} T^i x - \sum_{i=0}^{n} T^i x\|}{n+2} + \frac{1}{(n+2)(n+1)}\|\sum_{i=0}^{n} T^i x\| \\
&\le \frac{1}{n+2}\|T^{n+1}x\| + \frac{1}{(n+2)(n+1)}\sum_{i=0}^{n}\|T^i x\| \\
&\le \frac{1}{n+2}M + \frac{1}{n+2}M = \frac{2M}{n+2} \to 0.
\end{aligned}$$

Since $X$ has a uniformly Gâteaux differentiable norm, the duality mapping is uniformly continuous on bounded subsets of $X$ from the strong topology of $X$ to the weak star topology of $X^*$ (see e.g.,[7]). By Lemma 5 and Theorem 3.2 of [10], it is easy to prove the following theorem.

THEOREM 6. Let $X$ be a reflexive and strictly convex Banach space which has uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $X$, $T : C \to C$ a non-expansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with contraction constant $\alpha$. For any given $z_0 \in C$, let $\{z_n\}$ be the iterative sequence defined by (3) and $\lim_{n\to\infty} t_n = 0$. Suppose that $T$ satisfies property (A). Then $\{z_n\}$ converges strongly to some fixed point $p$ of $T$.

LEMMA 7. Let $C$ be a nonempty closed convex subset of a real Banach space $X$, $T$ be a non-expansive self-mapping of $C$ with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $[0, 1]$ and satisfy (i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$; (iii) $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Let the sequence $\{x_n\}$ be defined by (1) and $T_n = \frac{1}{n+1}\sum_{j=0}^{n} T^j$, then we have

(a) $\{x_n\}$ is bounded;

(b) $\lim_{n\to\infty} \|x_n - T_n x_n\| = 0$.

PROOF. (a). From Lemma 5(i) we have $T_n$ is non-expansive. We now show that the sequence $\{x_n\}$ is bounded. In fact, take $u \in F(T)$,

$$\begin{aligned}
\|x_{n+1} - u\| &= \|\alpha_n(f(x_n) - u) + \beta_n(x_n - u) + \gamma_n(T_n x_n - u)\| \\
&\leq \alpha_n\|f(x_n) - u\| + \beta_n\|x_n - u\| + \gamma_n\|T_n x_n - u\| \\
&\leq (1 - (1 - \alpha)\alpha_n)\|x_n - u\| + \alpha_n\|f(u) - u\|.
\end{aligned}$$

It implies by induction that

$$\|x_n - u\| \leq \max\left\{\|x_0 - u\|, \frac{1}{1-\alpha}\|f(u) - u\|\right\}$$

and $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{Tx_n\}$ and $\{T_n x_n\}$. Next we rewrite the iteration process (1) as follows

$$\begin{aligned}
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + \gamma_n T_n x_n \\
&= \beta_n x_n + (1 - \beta_n)\left[\frac{\alpha_n}{1-\beta_n} f(x_n) + \frac{\gamma_n}{1-\beta_n} T_n x_n\right].
\end{aligned}$$

Thus, if we set $y_n = \tilde{\gamma}_n f(x_n) + (1 - \tilde{\gamma}_n) T_n x_n$, where $\tilde{\gamma}_n = \frac{\alpha_n}{1-\beta_n}$, then we get

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) y_n. \tag{5}$$

It is easy to check that $\{y_n\}$ is bounded.

Next we show that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. To see this, we calculate

$$\begin{aligned}
y_{n+1} - y_n &= \tilde{\gamma}_{n+1} f(x_{n+1}) + (1 - \tilde{\gamma}_{n+1}) T_{n+1} x_{n+1} - \tilde{\gamma}_n f(x_n) - (1 - \tilde{\gamma}_n) T_n x_n \\
&= \tilde{\gamma}_{n+1}(f(x_{n+1}) - f(x_n)) + (1 - \tilde{\gamma}_{n+1})(T_{n+1} x_{n+1} - T_{n+1} x_n) \\
&\quad + (1 - \tilde{\gamma}_{n+1})(T_{n+1} x_n - T_n x_n) + (\tilde{\gamma}_{n+1} - \tilde{\gamma}_n)(f(x_n) - T_n x_n).
\end{aligned}$$

It follows that

$$\begin{aligned}
\|y_{n+1} - y_n\| &\leq \alpha\tilde{\gamma}_{n+1}\|x_{n+1} - x_n\| + (1 - \tilde{\gamma}_{n+1})\|T_{n+1} x_n - T_n x_n\| \\
&\quad + (1 - \tilde{\gamma}_{n+1})\|x_{n+1} - x_n\| + |\tilde{\gamma}_{n+1} - \tilde{\gamma}_n|\|f(x_n) - T_n x_n\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|y_{n+1} - y_n\| &- \|x_{n+1} - x_n\| \\
&\leq (\alpha - 1)\tilde{\gamma}_{n+1}\|x_{n+1} - x_n\| + |\tilde{\gamma}_{n+1} - \tilde{\gamma}_n|\|f(x_n) - T_n x_n\| \\
&\quad + (1 - \tilde{\gamma}_{n+1})\|T_{n+1} x_n - T_n x_n\|.
\end{aligned}$$

Thus, we have from (ii) and Lemma 5(ii) that

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0.$$

Apply Lemma 3 to get $\|y_n - x_n\| = 0$. Again by Eq. (5) we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} (1 - \beta_n)\|y_n - x_n\| = 0.$$

This implies from (1) that

$$\begin{aligned}\|x_n - T_n x_n\| &\leq \|x_{n+1} - x_n\| + \|x_{n+1} - T_n x_n\| \\ &\leq \|x_{n+1} - x_n\| + \alpha_n \|f(x_n) - T_n x_n\| + \beta_n \|x_n - T_n x_n\|.\end{aligned}$$

It follows that

$$\|x_n - T_n x_n\| \leq \frac{1}{1-\beta_n}\|x_{n+1} - x_n\| + \frac{\alpha_n}{1-\beta_n}\|f(x_n) - T_n x_n\| \to 0. \tag{6}$$

We give an example concerning $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$.

EXAMPLE 8. For each $n \geq 1$, we set $\alpha_n = \frac{1}{n+4}$, $\gamma_n = 1 - \alpha_n - \beta_n$ and

$$\beta_n = \begin{cases} \frac{1}{3} + \frac{1}{n+5} & \text{if } n \text{ is odd} \\ \frac{1}{4} + \frac{1}{n+6} & \text{if } n \text{ is even} \end{cases}.$$

Then $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ satisfying the assumption of Lemma 7.

THEOREM 9. Let $X$ be a reflexive and strictly convex Banach space which has uniformly Gâteaux differentiable norm, $C$ a nonempty closed convex subset of $X$, $T : C \to C$ a non-expansive mapping with $F(T) \neq \emptyset$ and $f : C \to C$ be a contraction with contraction constant $\alpha$. Let $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ be three real sequences in $[0,1]$ and satisfy (i) $\alpha_n + \beta_n + \gamma_n = 1$; (ii) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$; (iii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$. Let the sequence $\{x_n\}$ be defined by (1). Suppose $T$ satisfies property (A), then $\{x_n\}$ converges strongly to fixed point of $T$.

PROOF. By Lemma 7, We have the following assertions:

(I) $\{x_n\}$ is bounded, so are $\{f(x_n)\}$ and $\{T_n x_n\}$;

(II) $\lim_{n\to\infty} \|T_n x_n - x_n\| = 0$.

We show that

$$\limsup_{n\to\infty} \langle p - f(p), j(p - x_n)\rangle \leq 0. \tag{7}$$

Indeed we can write $z_m - x_n = t_m(f(z_m) - x_n) + (1 - t_m)(T_m z_m - x_n)$. Putting

$$\begin{aligned}P_n(m) &= (\|T_m x_n - T_n x_n\| + \|T_n x_n - x_n\|)(\|T_m x_n - T_n x_n\| \\ &\quad +\|T_n x_n - x_n\| + 2\|z_m - x_n\|),\end{aligned}$$

then we have from (II) that

$$\begin{aligned}\limsup_{n\to\infty} \|T_m x_n - T_n x_n\| &\leq \limsup_{n\to\infty} \|T_{n-m} x_n - x_n\| \\ &\leq \limsup_{n\to\infty} \|T_n x_n - x_n\| = 0.\end{aligned}$$

It follows that $\limsup_{n\to\infty} P_n(m) = 0$ and using Lemma 1, we obtain

$$\begin{aligned}\|z_m - x_n\|^2 &\leq (1-t_m)^2\|T_m z_m - x_n\|^2 + 2t_m\langle f(z_m) - x_n, J(z_m - x_n)\rangle \\ &\leq (1-t_m)^2(\|T_m z_m - T_m x_n\| + \|T_m x_n - T_n x_n\| + \|T_n x_n - x_n\|)^2 \\ &\quad +2t_m\langle f(z_m) - x_n, J(z_m - x_n)\rangle \\ &\leq (1-t_m)^2\|z_m - x_n\|^2 + P_n(m) + 2t_m\|z_m - x_n\|^2 \\ &\quad +2t_m\langle f(z_m) - z_m, J(z_m - x_n)\rangle.\end{aligned}$$

The last inequality implies

$$\langle z_m - f(z_m), J(z_m - x_n)\rangle \le \frac{t_m}{2}\|z_m - x_n\|^2 + \frac{1}{2t_m}P_n(m).$$

This implies that

$$\limsup_{n\to\infty}\langle z_m - f(z_m), J(z_m - x_n)\rangle \le M\frac{t_m}{2}$$

where $M > 0$ is a constant such that $M \ge \|z_m - x_n\|^2$ for all $m, n \ge 1$. Taking the $\limsup$ as $m \to \infty$, by Theorem 6 we obtain (7).

Finally we show that $x_n \to p$. Apply Lemma 1 and (1) to get

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \|\alpha_n(f(x_n) - p) + \beta_n(x_n - p) + \gamma_n(T_n x_n - p)\|^2 \\
&\le \|\beta_n(x_n - p) + \gamma_n(T_n x_n - p)\|^2 + 2\alpha_n\langle f(x_n) - p, J(x_{n+1} - p)\rangle \\
&\le (1 - \alpha_n)^2\|x_n - p\|^2 + 2\alpha_n\langle f(x_n) - f(p), J(x_{n+1} - p)\rangle \\
&\quad +2\alpha_n\langle f(p) - p, J(x_{n+1} - p)\rangle \\
&\le (1 - \alpha_n)^2\|x_n - p\|^2 + 2\alpha\alpha_n\|x_n - p\|\|x_{n+1} - p\| \\
&\quad +2\alpha_n\langle f(p) - p, J(x_{n+1} - p)\rangle \\
&\le (1 - \alpha_n)^2\|x_n - p\|^2 + \alpha\alpha_n(\|x_n - p\|^2 + \|x_{n+1} - p\|^2) \\
&\quad +2\alpha_n\langle f(p) - p, J(x_{n+1} - p)\rangle.
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le \frac{1 - \alpha_n(2 - \alpha)}{1 - \alpha\alpha_n}\|x_n - p\|^2 + \frac{\alpha_n^2}{1 - \alpha\alpha_n}\|x_n - p\|^2 \\
&\quad +\frac{2\alpha_n}{1 - \alpha\alpha_n}\langle f(p) - p, J(x_{n+1} - p)\rangle \\
&\le \left(1 - \frac{2(1 - \alpha)\alpha_n}{1 - \alpha\alpha_n}\right)\|x_n - p\|^2 + \frac{\alpha_n^2}{1 - \alpha\alpha_n}\|x_n - p\|^2 \\
&\quad +\frac{2\alpha_n}{1 - \alpha\alpha_n}\langle f(p) - p, J(x_{n+1} - p)\rangle \\
&\le (1 - \tilde{t}_n)\|x_n - p\|^2 + \frac{\alpha_n}{1 - \alpha\alpha_n}(\alpha_n M + 2\tilde{\gamma}_{n+1})
\end{aligned}$$

where $\tilde{t}_n = \frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$, $M > \sup_n \|x_n - p\|^2$ and $\tilde{\gamma}_{n+1} = \max\{\langle f(p) - p, J(x_{n+1} - p)\rangle, 0\}$. It is easily seen that $\lim_{n\to\infty} \tilde{\gamma}_{n+1} = 0$. Apply Lemma 2 to conclude that $x_n \to p$.

**Acknowledgment.** The authors would like to thank Prof. Nikhat Bano and the anonymous referee for his valuable suggestions which helps to improve this manuscript.

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