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An Answer To The Conjecture Of Satnoianu*

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Abstract

In this short paper, we obtain an answer to the conjecture of Satnoianu by a simpler method in the view of probability theory. The conditions of our results are independent with some known answers.

1 Introduction

In [2], Mazur proposed the open problem: *if a, b, c are positive real numbers such that $abc > 2^9$, then*

$$\frac{1}{\sqrt{1+a}} + \frac{1}{\sqrt{1+b}} + \frac{1}{\sqrt{1+c}} \geq \frac{3}{\sqrt{1+\sqrt[3]{abc}}}. \tag{1}$$

In fact, in 2001, Satnoianu [3] has studied the following inequality

$$\sum_{cyclic} \frac{a}{\sqrt{a^2+\lambda bc}} \geq \frac{3}{\sqrt{1+\lambda}} \quad (a, b, c > 0, \lambda \geq 8). \tag{2}$$

In addition, Satnoianu proposed the following inequality as a conjecture

$$\sum_{i=1}^{n} \left(\frac{x_i^{n-1}}{x_i^{n-1} + \lambda \prod_{k \neq i} x_k} \right)^{\frac{1}{n-1}} \geq n(1+\lambda)^{-\frac{1}{n-1}}. \tag{3}$$

Shortly after the proposed conjecture, Janous [1] gave the proof of the inequality (3) by means of Lagrange's method of multipliers and Satnoianu [4] obtained a generalized version of inequality (3) as follows

$$\sum_{i=1}^{n} \left(\frac{x_i^{n-1}}{\alpha x_i^{n-1} + \beta \prod_{k \neq i} x_k} \right)^{\frac{1}{n-1}} \geq n(\alpha+\beta)^{-\frac{1}{n-1}}, \tag{4}$$

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where $n \geq 2$, $x_i > 0$, $i = 1, 2, \ldots, n$, $\alpha, \beta > 0$ and $\beta \geq (n^{n-1} - 1)\alpha$. Recently, Wu [5] established the following more generalized inequality

$$\sum_{i=1}^{n} \left(\frac{x_i^q}{\alpha x_i^q + \beta \prod_{k=1}^{n} x_k^{q/n}} \right)^{\frac{1}{p}} \geq n(\alpha + \beta)^{-\frac{1}{p}}, \tag{5}$$

where $\alpha, \beta, x_i (i = 1, 2, \ldots, n)$ are positive real numbers, $q \in \mathbb{R}$, and $p < 0$, or $p > 0$ with $\beta \geq (n^{\max\{p,1\}} - 1)\alpha$.

If we rewrite the inequality (5) as

$$\frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{\alpha + \beta \exp\left\{ \frac{1}{n} \sum_{k=1}^{n} \log x_k^q - \log x_i^q \right\}} \right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}, \tag{6}$$

then it is easy to see that (6) is equivalent to

$$E \left(\frac{X}{\alpha X + \beta \exp\{E \log X\}} \right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}, \tag{7}$$

where X is a random variable taking values $x_1^q, x_2^q, \ldots, x_n^q$ with the probability $P(X = x_i^q) = \frac{1}{n}$ and $E(X)$ denotes the mathematical expectation of X. In fact, X can be an any positive random variable. Hence we could generalize the conjecture of Satnoianu as: *"Under what conditions does the inequality (7) holds?"*

2 Main Results

Before our works, we need give the following useful

LEMMA 1. Let $f(x) = (a + be^x)^p$, where $a, b > 0$, $x \in \mathbb{R}$. If $p > 0$ or if $p < 0$ with $pbe^x + a \leq 0$, then $f(x)$ is a convex function.

PROOF. The method is elementary. Since a twice differentiable function of one variable is convex on an interval if and only if its second derivative is non-negative and

$$f'(x) = pb(a + be^x)^{p-1} e^x,$$

$$\begin{aligned} f''(x) &= p(p-1)b^2(a + be^x)^{p-2} e^{2x} + pb(a + be^x)^{p-1} e^x \\ &= pbe^x (a + be^x)^{p-2} [(p-1)be^x + (a + be^x)] \\ &= pbe^x (a + be^x)^{p-2} [pbe^x + a], \end{aligned}$$

the desired result is easy to be obtained.

PROPOSITION 1. Let random variable $X > 0$ a.e. and $\alpha, \beta > 0$. If $p < 0$ or if $p > 0$ with $X \leq \beta e^{E \log X} / (\alpha p)$ a.e., then we have

$$E \left(\frac{X}{\alpha X + \beta \exp\{E \log X\}} \right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}. \tag{8}$$

PROOF. Let $Y = -\log X$, then (8) is equivalent to

$$E\left(\frac{1}{\alpha + \beta e^{-EY} e^{Y}}\right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}. \tag{9}$$

By Lemma 1. and Jensen's inequality, the proof is easy to be obtained.

From the above proposition, we have the following result and the proof is easy.

THEOREM 1. Let $\alpha, \beta > 0$ and X be a discrete random variable taking positive numbers $x_1, x_2, \ldots, x_n$ with $P(X = x_i) = a_i$, where $\sum_{i=1}^{n} a_i = 1$. In addition, let $M = \max\{x_i, 1 \leq i \leq n\}$ and $m = \min\{x_i, 1 \leq i \leq n\}$. If $p < 0$ or if $p > 0$ with $M/m \leq \beta/(\alpha p)$, then we have

$$\sum_{i=1}^{n} a_i \left(\frac{x_i}{\alpha x_i + \beta \prod_{k=1}^{n} x_k^{a_i}}\right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}. \tag{10}$$

In particular, if $a_1 = a_2 = \cdots = a_n = \frac{1}{n}$, we have

$$\sum_{i=1}^{n} \left(\frac{x_i}{\alpha x_i + \beta \prod_{k=1}^{n} x_k^{1/n}}\right)^{\frac{1}{p}} \geq n(\alpha + \beta)^{-\frac{1}{p}}. \tag{11}$$

REMARK 1. By comparing the conditions of Theorem 1. with the ones of Wu in [5], we find that these assumptions are independent each other. In fact, the only difference is between "$M/m \leq \beta/(\alpha p)$" and "$\beta \geq (n^{\max\{p,1\}} - 1)\alpha$", from that we can not judge which condition is weaker than the other.

REMARK 2. For the infinite sequence $\{x_i\}_{i=1}^{\infty}$, let $\sum_{i=1}^{\infty} a_i = 1$, $M = \sup_{i \geq 1} x_i < \infty$ and $m = \inf_{i \geq 1} x_i > 0$, then by the same discussions as Theorem 1., we have

$$\sum_{i=1}^{\infty} a_i \left(\frac{x_i}{\alpha x_i + \beta \prod_{k=1}^{\infty} x_k^{a_i}}\right)^{\frac{1}{p}} \geq (\alpha + \beta)^{-\frac{1}{p}}. \tag{12}$$

The following result is the integral form of the conjecture of Satnoianu.

THEOREM 2. Let $\alpha, \beta > 0$ and X be a positive continuous random variable on $(0, \infty)$ with the probability density function $f(x)$. If $p < 0$ or if $p > 0$ with $X \leq \beta e^{E \log X}/(\alpha p)$ a.e., then we have

$$\int_0^{\infty} \left(\frac{x}{\alpha x + \beta \exp\left\{\int_0^{\infty} \log x f(x) dx\right\}}\right)^{\frac{1}{p}} f(x) dx \geq (\alpha + \beta)^{-\frac{1}{p}}. \tag{13}$$

In particular, if X possesses uniform distribution on the support interval $[a, b]$, i.e., the probability density function of X is equal to $(b - a)^{-1}, x \in [a, b]$ and zero elsewhere. Then if $b/a \leq \beta/(\alpha p)$, then we have

$$\frac{1}{b - a} \int_a^b \left(\frac{x}{\alpha x + \beta \exp\left\{\frac{1}{b-a} \int_a^b \log x dx\right\}}\right)^{\frac{1}{p}} dx \geq (\alpha + \beta)^{-\frac{1}{p}}. \tag{14}$$

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