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A Note On Some Fixed Point Results*

Ivan D. Aranđelović[†], Dojčin S. Petković[‡]

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Abstract

In [3] M. Aamri and D. El Moutawakil proved two general common fixed point theorems for self-mappings on semi-metric space. Here we show that many fixed point theorems which use contractive conditions of integral type can be obtained as corollaries.

1 Introduction

Contraction mapping principle, formulated and proved in the Ph. D. dissertation of S. Banach in 1920 which was published in 1922, is one of the most important theorems in classical functional analysis because it gives:

1. the existence and uniqueness of fixed point,
2. method for obtaining approximative fixed points, and
3. error estimates for approximative fixed point obtained by 2.

There are many generalizations and partial generalizations of the Banach principle. One such generalization is formulated in semi-metric spaces initiated by M. Fréchet, K. Menger [11], E. W. Chittenden [5] and W. A. Wilson [16]. In [6] Cicchese introduced the notion of a contraction mapping in semi-metric spaces and proved the first fixed point theorem for this class of spaces. Further fixed point results for this class of spaces were obtained by J. Jachymski, J. Matkowski and T. Swaitkowski [10], T. L. Hicks, B. E. Rhoades [8], M. Aamri and D. El Moutawakil [3], J. Zhu, Y. J. Cho, S. M. Kang [18], D. Miheţ [12], M. Imdad, J. Ali and L. Khan [9], A. Aliouche [1], etc.

In 2002 A. Branciari [4] introduced the notion of contractions of integral type and proved fixed point theorem for this class of mappings. Further results on this class of mappings were obtained by B. E. Rhoades [14], A. Aliouche [1, 2], A. Djoudi and F. Merghadi [7] and many others. Zhang [17] gave new generalized contractive type condition in which the integral operator is replaced by a monotone nondecreasing function.

In [3] M. Aamri and D. El Moutawakil proved two general common fixed point theorems for self-mappings on semi-metric space. We intend to show that many fixed point theorems which used contractive conditions of integral type can be obtained as corollaries of these results.

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†University of Belgrade, Faculty of Mechanical Engineering, Kraljice Marije 16, 11000 Beograd, Serbia,e-mail: iarandjelovic@mas.bg.ac.rs

‡University of Priština, Faculty of Science and Mathematics, Knjaza Miloša 7, 28220 Kosovska Mitrovica, Serbia.

2 Preliminary Notes

Let X be a non-empty set and $d : X^2 \to [0, \infty)$. (X, d) is semi-metric space (symmetric space) if and only if it satisfies:

(W1) $d(x, y) = 0$ if and only if $x = y$; and

(W2) $d(x, y) = d(y, x)$ if and only if $x = y$ for any $x, y \in X$.

Let (X, d) be a semi-metric (symmetric) space, $r > 0$ and $x \in X$, let $B(x, r) = \{y \in X : \ d(x, y) < r\}$. Let τ be the weakest topology on X such that the family $\{B(x, r) : \ x \in X, \ r \in [0, \infty)\}$ is the base for τ. Note that for every $\{x_n\} \subseteq X$ and $x \in X$, $\lim d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology τ. A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence, if for every given $\varepsilon > 0$, there exists a positive integer n_0 such that $d(x_m, x_n) < \varepsilon$ for all $m, n \geq n_0$. A semi-metric space (X, d) is complete if and only if each its Cauchy sequence is convergent.

Let (X, d) be a semi-metric (symmetric) space. Then:

- (X, d) satisfies the property (W3) if and only if $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$ imply $x = y$;
- (X, d) satisfies the property (W4) if and only if $\lim d(x_n, x) = 0$ and $\lim d(x_n, y_n) = 0$ imply $\lim d(y_n, x) = 0$;
- (X, d) satisfies the property (HE) if and only if $\lim d(x_n, x) = 0$ and $\lim d(y_n, x) = 0$ imply $\lim d(x_n, y_n) = 0$;
- (X, d) satisfies the property (W) if and only if $\lim d(x_n, y_n) = 0$ and $\lim d(y_n, z_n) = 0$ imply $\lim d(x_n, z_n) = 0$.

All this conditions can be used as partial replacement for the triangle inequality. (W3) and (W4) were introduced by Wilson [16], (HE) by M. Aamri and D. El Moutawakil [3] and (W) by D. Miheţ [12]. Note that $(W) \Rightarrow (W4) \Rightarrow (W3)$ and $(W) \Rightarrow (HE)$.

Let X be a nonempty set and $F, G : X \to X$ arbitrary mapping. $x \in X$ is a fixed point of F if $x = Fx$. $y \in X$ is a coincidence point for F and G if and only if $Fy = Gy$.

Let (X, d) be a semi-metric space and $F, G : X \to X$. Then:

- F and G are said to be compatible if and only if
$$\lim d(FGx_n, GFx_n) = 0$$
whenever $\{x_n\}$ is a sequence in X such that
$$\lim d(Fx_n, t) = \lim d(Gx_n, t) = 0$$
for some $t \in X$;
- F and G are said to be weakly compatible if and only if they commute at their coincidence point; i.e., if $Fx = Gx$ then $FGx = GFx$;

- F and G are said to satisfy property (E.A) if there exists a sequence $\{x_n\}$ such that
$$\lim Fx_n = \lim Gx_n = t$$
for some $t \in X$.

By Φ we denote the set of all real functions $\varphi : [0, \infty) \to [0, \infty)$ with the following properties: (a) $\varphi(0) = 0$; (b) $\varphi(r) < r$ for all $r > 0$; (c) $\overline{\lim}_{t \to r+} \varphi(t) < r$ for any $r > 0$.

LEMMA 1. (M. Tasković [15]) Let $\varphi \in \Phi$, $x_0 > 0$ and $\{x_n\}$ be a sequence defined by $x_n = \varphi^n(x_0)$. Then $\lim x_n = 0$.

By Λ we denote the set of all nonnegative, Lebesgue-integrable, real functions $\lambda : [0, \infty) \to [0, \infty)$ such that
$$0 < \int_0^{\varepsilon} \lambda(t)dt < \infty \text{ for all } \varepsilon > 0.$$

By $\mathcal{F}$ we denote the set of all continuous, monotone nondecreasing, real functions $F : [0, \infty) \to [0, \infty)$ such that $F(x) = 0$ if and only if $x = 0$.

In [17] it was proved:

LEMMA 2 (X. Zhang [17]). Let $F \in \mathcal{F}$ and $\{\varepsilon_n\} \subseteq [0, \infty)$. Then from $F(\varepsilon_n) \to 0$ follows that $\varepsilon_n \to 0$.

In [3] M. Aamri and D. El Moutawakil proved the following common fixed point theorems.

THEOREM 1 (M. Aamri and D. El Moutawakil [3]). Let (X, d) be a semi-metric (symmetric) space which satisfies properties (W3) and (HE). Let $\varphi \in \Phi$ and let $A, B : X \to X$ be self-mappings of X such that:

1) $d(Ax, Ay) \le \varphi(\max\{d(Bx, By), d(Bx, Ay), d(Ay, By)\})$ for any $x, y \in X$;
2) A and B are weakly compatible;
3) A and B satisfy the property (E.A);
4) $AX \subseteq BX$.

If the range of one of the mappings A or B is a complete subspace of X, then A and B have a unique common fixed point.

THEOREM 2 (M. Aamri and D. El Moutawakil [3]). Let (X, d) be a semi-metric (symmetric) space which satisfies properties (W4) and (HE). Let $\varphi \in \Phi$ and let $A, B, S, T : X \to X$ be self-mappings of X such that:

1) $d(Ax, By) \le \varphi(\max\{d(Sx, Ty), d(Sx, By), d(By, Ty)\})$ for any $x, y \in X$;
2) (A, T) and (B, S) are weakly compatible;
3) (A, S) or (B, T) satisfies the property (E.A);
4) $AX \subseteq TX$ and $BX \subseteq SX$.

If the range of one of the mappings A, B, S or T is a complete subspace of X, then A, B, S and T have a unique common fixed point.

In this paper we present some new applications of these theorems.

3 Results

We need the following Lemma.

LEMMA 3. Let (X, d) be a semi-metric space, $x \in X$, $\{x_n\} \subseteq X$ and $F \in \mathcal{F}$. Define $d^* : X^2 \to [0, \infty)$ by

$$d^*(x, y) = F(d(x, y)), \text{ for any } x, y \in X.$$

Then:
1) (X, d^*) is semi-metric space;
2) $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if it is a Cauchy sequence in (X, d^*);
3) $\lim d(x_n, x) = 0$ if and only if $\lim d^*(x_n, x) = 0$.

PROOF. To see 1), note that (W1) follows from $d(x, y) = 0 \Leftrightarrow F(d(x, y)) = 0$, and (W2) follows from $F(d(x, y)) = F(d(y, x))$.

Next, let $\{x_n\}$ be a Cauchy sequence in (X, d). Then

$$\lim_{n,k\to\infty} d(x_{n+k}, x_n) = 0,$$

which implies

$$\lim_{n,k\to\infty} F(d(x_{n+k}, x_n)) = F(0) = 0$$

because F is continuous. So $\{x_n\}$ is a Cauchy sequence in (X, d^*).

Let $\{x_n\}$ be a Cauchy sequence in (X, d^*). Then

$$\lim_{n,k\to\infty} F(d(x_{n+k}, x_n)) = 0.$$

By Lemma 2 we get that

$$\lim_{n,k\to\infty} d(x_{n+k}, x_n) = 0,$$

which implies that $\{x_n\}$ is a Cauchy sequence in (X, d).

Finally, let $\lim d(x_n, x) = 0$. It follows that $\lim F(d(x_n, x)) = F(0) = 0$, because F is continuous. Let $\lim d^*(x_n, x) = 0$. By Lemma 2 it follows that $\lim d(x_n, x) = 0$.

Now we shall prove our next result.

THEOREM 3. Let (X, d) be a semi-metric space and $F \in \mathcal{F}$. Define $d^* : X^2 \to [0, \infty)$ by $d^*(x, y) = F(d(x, y))$ for any $x, y \in X$. Then
- (X, d) satisfies the property (W3) if and only if (X, d^*) satisfies this property;
- (X, d) satisfies the property (W4) if and only if (X, d^*) satisfies this property;
- (X, d) satisfies the property (HE) if and only if (X, d^*) satisfies this property;
- (X, d) satisfies the property (W) if and only if (X, d^*) satisfies this property;
- (X, d) is complete if and only if (X, d^*) is complete.

PROOF. Let (X, d) be a semi-metric space which satisfies the property (W3). Let $\lim F(d(x_n, x)) = 0$ and $\lim F(d(x_n, y)) = 0$. By Lemma 2 it follows that $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$. By (W3) we get that $x = y$. If (X, d^*) satisfies (W3), then from $\lim d(x_n, x) = 0$ and $\lim d(x_n, y) = 0$ it follows that $\lim F(d(x_n, x)) = F(0) = 0$ and $\lim F(d(x_n, y)) = F(0) = 0$, because F is continuous. So $x = y$, because (X, d^*) satisfies (W3).

Let (X,d) be a semi-metric space which satisfies (W4). Let $\lim F(d(x_n,x))=0$ and $\lim F(d(x_n,y_n))=0$. By Lemma 2 it follows that $\lim d(x_n,x)=0$ and $\lim d(x_n,y_n)=0$. By (W4) we get that $\lim d(y_n,x)=0$, which implies $\lim F(d(y_n,x))=F(0)=0$, because F is continuous. If (X,d^*) satisfies (W4), then from $\lim d(x_n,x)=0$ and $\lim d(x_n,y_n)=0$ it follows that $\lim F(d(x_n,x))=F(0)=0$ and $\lim F(d(x_n,y_n))=F(0)=0$, because F is continuous. So $\lim F(d(y_n,x))=0$, because (X,d^*) satisfies (W4).

Let (X,d) be a semi-metric space which satisfies (HE). Let $\lim F(d(x_n,x))=0$ and $\lim F(d(y_n,x))=0$. By Lemma 2 it follows that $\lim d(x_n,x)=0$ and $\lim d(y_n,x)=0$. By (HE) we get that $\lim d(x_n,y_n)=0$, which implies $\lim F(d(x_n,y_n))=F(0)=0$, because F is continuous. If (X,d^*) satisfies (HE), then from $\lim d(x_n,x)=0$ and $\lim d(y_n,x)=0$ it follows that $\lim F(d(x_n,x))=F(0)=0$ and $\lim F(d(y_n,x))=F(0)=0$, because F is continuous. So $\lim F(d(x_n,y_n))=0$, because (X,d^*) satisfies (HE).

Let (X,d) be a semi-metric space which satisfies (W). Let $\lim F(d(x_n,y_n))=0$ and $\lim F(d(y_n,z_n))=0$. By Lemma 2 it follows that $\lim d(x_n,y_n)=0$ and $\lim d(y_n,z_n)=0$. By (W) we get that $\lim d(x_n,z_n)=0$, which implies $\lim F(d(x_n,z_n))=F(0)=0$, because F is continuous. If (X,d^*) satisfies (W), then from $\lim d(x_n,y_n)=0$ and $\lim d(y_n,z_n)=0$ it follows that $\lim F(d(x_n,y_n))=F(0)=0$ and $\lim F(d(y_n,z_n))=F(0)=0$, because F is continuous. So $\lim F(d(x_n,z_n))=0$, because (X,d^*) satisfies (W).

The last statement of this theorem (equi-completeness of (X,d) and (X,d^*)) follows from Lemma 3.2.

From Theorem 3 it follows:

THEOREM 4. Let (X,d) be a metric space and $F\in\mathcal{F}$. Define $d^*: X^2\to[0,\infty)$ by $d^*(x,y)=F(d(x,y))$ for any $x,y\in X$. Then (X,d^*) is a semi-metric space which satisfies the property (W).

PROOF. From $\lim F(d(x_n,y_n))=0$ and $\lim F(d(y_n,z_n))=0$ by Lemma 2 it follows that $\lim d(x_n,y_n)=0$ and $\lim d(y_n,z_n)=0$. Hence $\lim d(x_n,z_n)=0$, because

$$d(x_n,z_n)\le d(x_n,y_n)+d(y_n,z_n) \text{ for each } n.$$

So $\lim F(d(x_n,z_n))=0$, because F is continuous.

Now we shall prove our next result.

THEOREM 5. Let (X,d) be a semi-metric (symmetric) space which satisfies properties (W3) and (HE). Let $\varphi\in\Phi$, $F\in\mathcal{F}$ and let $A,B: X\to X$ be self-mappings of X such that:

1) $F(d(Ax,Ay))\le\varphi(F(\max\{d(Bx,By),d(Bx,Ay),d(Ay,By)\}))$ for any $x,y\in X$;
2) A and B are weakly compatible;
3) A and B satisfy the property (E.A);
4) $AX\subseteq BX$.

If the range of one of the mappings A or B is a complete subspace of X, then A and B have a unique common fixed point.

PROOF. Define $d^* : X^2 \to [0,\infty)$ by $d^*(x,y) = F(d(x,y))$, for any $x, y \in X$, we have

$$\begin{aligned} & F(\max\{d(Sx,Ty), d(Sx,By), d(By,Ty)\}) \\ = \ & \max\{F(d(Sx,Ty)), F(d(Sx,By)), F(d(By,Ty))\}) \\ = \ & \max\{d^*(Sx,Ty), d^*(Sx,By), d^*(By,Ty)\}), \end{aligned}$$

because F is monotone nondecreasing function. Hence

$$d(Ax,By) \le \varphi(\max\{d^*(Sx,Ty), d^*(Sx,By), d^*(By,Ty)\}) \text{ for any } x, y \in X.$$

Therefore, the hypotheses of Theorem 1 are satisfied.

Let $\lambda \in \Lambda$. If in Theorem 5 F is defined by

$$F(x) = \int_0^x \lambda(t)dt, \text{ for any } x \ge 0,$$

then this Theorem reduces to the following result of A. Aliouche [1] - Corollary 1.

COROLLARY 1. (A. Aliouche [1]). Let (X,d) be a semi-metric (symmetric) space which satisfies properties (W3) and (HE). Let $\varphi \in \Phi$, $\lambda \in \Lambda$ and let $A, B : X \to X$ be self-mappings of X such that:
1) $\int_0^{d(Ax,Ay)} \lambda(t)dt \le \varphi(\int_0^{\max\{d(Bx,By),d(Bx,Ay),d(Ay,By)\}} \lambda(t)dt)$ for any $x, y \in X$;
2) A and B are weakly compatible;
3) A and B satisfy the property (E.A);
4) $AX \subseteq BX$.
If the range of one of the mappings A or B is a complete subspace of X, then A and B have a unique common fixed point.

Now we shall prove our next result.

THEOREM 6. Let (X,d) be a semi-metric (symmetric) space which satisfies properties (W4) and (HE). Let $\varphi \in \Phi$, $F \in \mathcal{F}$ and let $A, B, S, T : X \to X$ be self-mappings of X such that:
1) $F(d(Ax,By)) \le \varphi(F(\max\{d(Sx,Ty), d(Sx,By), d(By,Ty)\}))$ for any $x, y \in X$;
2) (A,T) and (B,S) are weakly compatible;
3) (A,S) or (B,T) satisfies the property (E.A);
4) $AX \subseteq TX$ and $BX \subseteq SX$.
If the range of one of the mappings A, B, S or T is a complete subspace of X, then A, B, S and T have a unique common fixed point.

PROOF. Define $d^* : X^2 \to [0,\infty)$ by $d^*(x,y) = F(d(x,y))$, for any $x, y \in X$. We have

$$\begin{aligned} & F(\max\{d(Sx,Ty), d(Sx,By), d(By,Ty)\}) \\ = \ & \max\{F(d(Sx,Ty)), F(d(Sx,By)), F(d(By,Ty))\}) = \\ = \ & \max\{d^*(Sx,Ty), d^*(Sx,By), d^*(By,Ty)\}), \end{aligned}$$

because F is monotone nondecreasing function. Hence

$$d(Ax,By) \le \varphi(\max\{d^*(Sx,Ty), d^*(Sx,By), d^*(By,Ty)\}) \text{ for any } x, y \in X.$$

Therefore, the hypotheses of Theorem 2 are satisfied.

Let $\lambda \in \Lambda$. If in Theorem 6 F is defined by

$$F(x) = \int_0^x \lambda(t)dt, \text{ for any } x \geq 0,$$

then this Theorem reduces to the following result of A. Aliouche [1] - Theorem 1.

COROLLARY 2 (A. Aliouche [1]). Let (X, d) be a semi-metric (symmetric) space which satisfies properties (W4) and (HE). Let $\varphi \in \Phi$, $\lambda \in \Lambda$ and let $A, B, S, T : X \to X$ be self-mappings of X such that:

1) $\int_0^{d(Ax,By)} \lambda(t)dt \leq \varphi(\int_0^{\max\{d(Sx,Ty),d(Sx,By),d(By,Ty)\}} \lambda(t)dt)$ for $x, y \in X$;
2) (A, T) and (B, S) are weakly compatibles;
3) (A, S) or (B, T) satisfies the property (E.A);
4) $AX \subseteq TX$ and $BX \subseteq SX$.

If the range of one of the mappings A, B, S or T is a complete subspace of X, then A, B, S and T have a unique common fixed point.

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