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Some Sharp Simpson Type Inequalities And Applications*

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Abstract

Some sharp Simpson type inequalities are proved. Applications in numerical integration are also considered.

1 Introduction

Given a real function of a real variable, let us write

$$f(\alpha|\beta) := f(\alpha) + 4f\left(\frac{\alpha+\beta}{2}\right) + f(\beta).$$

In [1], Ujević proved the following interesting sharp classical Simpson type inequality.

THEOREM 1. Let $f : [a, b] \to \mathbf{R}$ be an absolutely continuous function whose derivative $f' \in L_2(a, b)$. Then

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{6} f(a|b)\right| \le \frac{(b-a)^{\frac{3}{2}}}{6}\sqrt{\sigma(f')}, \tag{1}$$

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \frac{1}{b-a}\left(\int_a^b f(t)\,dt\right)^2 \tag{2}$$

and

$$\|f\|_2 := \left[\int_a^b f^2(t)\,dt\right]^{\frac{1}{2}}.$$

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Inequality (1) is sharp in the sense that the constant $\frac{1}{6}$ cannot be replaced by a smaller one.

An application in numerical integration has been given as

THEOREM 2. Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a, b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ and let the assumptions of Theorem 1 hold. Then

$$\left| \int_a^b f(x)\,dx - \frac{h}{6} \sum_{i=0}^{n-1} f(x_i|x_{i+1}) \right| \le \frac{b-a}{6n} \sigma_n(f) \le \frac{b-a}{6\sqrt{n}} \omega_n(f), \tag{3}$$

where $\sigma_n(f)$ and $\omega_n(f)$ are defined by

$$\sigma_n(f) = \sum_{i=0}^{n-1} \sqrt{\frac{b-a}{n} \|f'\|_2^2 - [f(x_{i+1}) - f(x_i)]^2},$$

and

$$\omega_n(f) = [(b-a)\|f'\|_2^2 - \frac{1}{n}(f(b) - f(a))^2]^{\frac{1}{2}}.$$

Obviously, the inequality (3) seems as if it is complicated and not convenient to obtain the error bounds. Recently in [2] the inequality (3) has been revised and improved as

$$\left| \int_a^b f(x)\,dx - \frac{h}{6} \sum_{i=0}^{n-1} f(x_i|x_{i+1}) \right| \le \frac{(b-a)^{\frac{3}{2}}}{6n} \sqrt{\sigma(f')}.$$

In this paper, we will further derive some sharp Simpson type inequalities. Applications in numerical integration are also considered.

2 Two More Sharp Classical Simpson Type Inequalities

We begin with the following result.

THEOREM 3. Let $f : [a, b] \to \mathbf{R}$ be such that f' is absolutely continuous on $[a, b]$ and $f'' \in L_2[a, b]$. Then we have

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{6} f(a|b) \right| \le \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{30}} \sqrt{\sigma(f'')}. \tag{4}$$

Inequality (4) is sharp in the sense that the constant $\frac{1}{12\sqrt{30}}$ cannot be replaced by a smaller one.

PROOF. Let us define the function

$$S_2(x) := \begin{cases} \frac{(x-a)^2}{2} - \frac{(b-a)(x-a)}{6}, & x \in \left[a, \frac{a+b}{2}\right], \\ \frac{(x-b)^2}{2} + \frac{(b-a)(x-b)}{6}, & x \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{5}$$

Integrating by parts, we obtain

$$\int_a^b S_2(x)f''(x)\,dx = \int_a^b f(x)\,dx - \frac{b-a}{6}f(a|b). \tag{6}$$

By elementary calculus, we have

$$\int_a^b S_2(x)\,dx = 0, \quad \int_a^b S_2^2(x)\,dx = \frac{(b-a)^5}{4320}. \tag{7}$$

Thus from (6), (7) and (2),we can easily get

$$\begin{aligned}
\left|\int_a^b f(x)\,dx - \frac{b-a}{6}f(a|b)\right| &= \left|\int_a^b S_2(x)f''(x)\,dx\right| \\
&= \left|\int_a^b S_2(x)[f''(x) - \frac{1}{b-a}\int_a^b f''(t)\,dt]dx\right| \\
&\le \left(\int_a^b S_2^2(x)\,dx\right)^{\frac{1}{2}} \left\{\int_a^b \left[f''(x) - \frac{f'(b)-f'(a)}{b-a}\right]^2 dx\right\}^{\frac{1}{2}} \\
&= \left[\frac{(b-a)^5}{4320}\right]^{\frac{1}{2}} \left\{\|f''\|_2^2 - \frac{[f'(b)-f'(a)]^2}{b-a}\right\}^{\frac{1}{2}} \\
&= \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{30}}\sqrt{\sigma(f'')}.
\end{aligned}$$

We now suppose that (4) holds with a constant $C > 0$ as

$$\left|\int_a^b f(x)\,dx - \frac{b-a}{6}f(a|b)\right| \le C(b-a)^{\frac{5}{2}}\sqrt{\sigma(f'')}. \tag{8}$$

We may find a function $f : [a,b] \to \mathbf{R}$ such that f' is absolutely continuous on $[a,b]$ as

$$f'(x) = \begin{cases} \frac{(x-a)^3}{6} - \frac{(b-a)(x-a)^2}{12} & \text{if } x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^3}{6} + \frac{(b-a)(x-b)^2}{12} & \text{if } x \in (\frac{a+b}{2}, b]. \end{cases}$$

It follows that

$$f''(x) = \begin{cases} \frac{(x-a)^2}{2} - \frac{(b-a)(x-a)}{6} & \text{if } x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^2}{2} + \frac{(b-a)(x-b)}{6} & \text{if } x \in (\frac{a+b}{2}, b]. \end{cases} \tag{9}$$

By (5)-(7) and (9), it is not difficult to find that the left-hand side of the inequality (8) becomes

$$L.H.S.(8) = \frac{(b-a)^5}{4320}, \tag{10}$$

and the right-hand side of the inequality (8) is

$$R.H.S.(8) = \frac{C(b-a)^5}{12\sqrt{30}}. \tag{11}$$

From (8), (10) and (11), we find that $C \geq \frac{1}{12\sqrt{30}}$, proving that the constant $\frac{1}{12\sqrt{30}}$ is the best possible in (4).

THEOREM 4. Let $f : [a, b] \to \mathbf{R}$ be such that f'' is absolutely continuous on $[a, b]$ and $f''' \in L_2[a, b]$. Then we have

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{6} f(a|b) \right| \leq \frac{(b-a)^{\frac{7}{2}}}{48\sqrt{105}} \sqrt{\sigma(f''')}. \tag{12}$$

Inequality (12) is sharp in the sense that the constant $\frac{1}{48\sqrt{105}}$ cannot be replaced by a smaller one.

PROOF. Let us define the function

$$S_3(x) := \begin{cases} \frac{(x-a)^3}{6} - \frac{(b-a)(x-a)^2}{12}, & x \in [a, \frac{a+b}{2}], \\ \frac{(x-b)^3}{6} + \frac{(b-a)(x-b)^2}{12}, & x \in (\frac{a+b}{2}, b]. \end{cases} \tag{13}$$

Integrating by parts, we obtain

$$\int_a^b S_3(x) f'''(x)\,dx = \frac{b-a}{6} f(a|b) - \int_a^b f(x)\,dx. \tag{14}$$

By elementary calculus, we have

$$\int_a^b S_3(x)\,dx = 0, \quad \int_a^b S_3^2(x)\,dx = \frac{(b-a)^7}{241920}. \tag{15}$$

Thus from (14), (15) and (2),we can easily get

$$\begin{aligned} \left| \int_a^b f(x)\,dx - \frac{b-a}{6} f(a|b) \right| &= \left| \int_a^b S_3(x) f'''(x)\,dx \right| \\ &= \left| \int_a^b S_3(x) \left[f'''(x) - \frac{1}{b-a} \int_a^b f'''(t)\,dt \right] dx \right| \\ &\leq \left(\int_a^b S_3^2(x)\,dx \right)^{\frac{1}{2}} \left\{ \int_a^b \left[f'''(x) - \frac{f''(b) - f''(a)}{b-a} \right]^2 dx \right\}^{\frac{1}{2}} \\ &= \left[\frac{(b-a)^7}{241920} \right]^{\frac{1}{2}} \left\{ \|f'''\|_2^2 - \frac{[f''(b) - f''(a)]^2}{b-a} \right\}^{\frac{1}{2}} \\ &= \frac{(b-a)^{\frac{7}{2}}}{48\sqrt{105}} \sqrt{\sigma(f''')}. \end{aligned}$$

We now suppose that (12) holds with a constant $C > 0$ as

$$\left| \int_a^b f(x)\,dx - \frac{b-a}{6} f(a|b) \right| \leq C(b-a)^{\frac{7}{2}} \sqrt{\sigma(f''')}. \tag{16}$$

We may find a function $f : [a, b] \to \mathbf{R}$ such that f'' is absolutely continuous on $[a, b]$ as

$$f''(x) = \begin{cases} \frac{(x-a)^4}{24} - \frac{(b-a)(x-a)^3}{36} & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \frac{(x-b)^4}{24} + \frac{(b-a)(x-b)^3}{36} & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases}$$

It follows that

$$f'''(x) = \begin{cases} \frac{(x-a)^3}{6} - \frac{(b-a)(x-a)^2}{12} & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \frac{(x-b)^3}{6} + \frac{(b-a)(x-b)^2}{12} & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{17}$$

By (13)-(15) and (17), it is not difficult to find that the left-hand side of the inequality (16) becomes

$$L.H.S.(16) = \frac{(b-a)^7}{241920}, \tag{18}$$

and the right-hand side of the inequality (16) is

$$R.H.S.(16) = \frac{C(b-a)^7}{48\sqrt{105}}. \tag{19}$$

From (16), (18) and (19), we find that $C \geq \frac{1}{48\sqrt{105}}$, proving that the constant $\frac{1}{48\sqrt{105}}$ is the best possible in (12).

REMARK 1. It should be noticed that the classical Simpson type inequalities (1), (4) and (12) have been appeared in [3] without the proofs of their sharpness but with some misprints.

3 Two Sharp Generalized Simpson Type Inequalities

In [4], we may find the identity

$$(-1)^n \int_a^b S_n(x) f^{(n)}(x)\, dx = \int_a^b f(x)dx - \frac{b-a}{6} f(a|b) + \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right), \tag{20}$$

where $[\frac{n-1}{2}]$ denotes the integer part of $\frac{n-1}{2}$ and $S_n(x)$ is the kernel given by

$$S_n(x) = \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{6(n-1)!} & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{6(n-1)!} & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{21}$$

By elementary calculus, it is not difficult to get

$$\int_a^b S_n(x)\, dx = \begin{cases} 0, & \text{n odd}, \\ -\frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^n}, & \text{n even}. \end{cases} \tag{22}$$

and

$$\int_a^b S_n^2(x)\,dx = \frac{(2n^3-11n^2+18n-6)(b-a)^{2n+1}}{9(4n^2-1)(n!)^2 2^{2n}}. \tag{23}$$

THEOREM 5. Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)}\in L_2[a,b]$ where n is an odd integer. Then we have

$$\begin{aligned}&\left|\int_a^b f(x)\,dx - \frac{b-a}{6}f(a|b) + \sum_{k=2}^{[\frac{n-1}{2}]}\frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}}f^{(2k)}\left(\frac{a+b}{2}\right)\right|\\ \leq\ & \frac{1}{3}\frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{2n^3-11n^2+18n-6}{4n^2-1}}\sqrt{\sigma(f^{(n)})}.\end{aligned} \tag{24}$$

Inequality (24) is sharp in the sense that the constant $\frac{1}{3}\frac{1}{2^n n!}\sqrt{\frac{2n^3-11n^2+18n-6}{4n^2-1}}$ cannot be replaced by a smaller one.

PROOF. From (20), (22), (23) and (2),we can easily get

$$\begin{aligned}&\left|\int_a^b f(x)\,dx - \frac{b-a}{6}f(a|b) + \sum_{k=2}^{[\frac{n-1}{2}]}\frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}}f^{(2k)}\left(\frac{a+b}{2}\right)\right|\\ =\ & \left|\int_a^b S_n(x)f^{(n)}(x)\,dx\right|\\ =\ & \left|\int_a^b S_n(x)\left[f^{(n)}(x)-\frac{1}{b-a}\int_a^b f^{(n)}(t)\,dt\right]\,dx\right|\\ \leq\ & \left(\int_a^b S_n^2(x)\,dx\right)^{\frac{1}{2}}\left(\int_a^b\left[f^{(n)}(x)-\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\right]^2\,dx\right)^{\frac{1}{2}}\\ =\ & \left(\frac{(2n^3-11n^2+18n-6)(b-a)^{2n+1}}{9(4n^2-1)(n!)^2 2^{2n}}\right)^{\frac{1}{2}}\left(\|f^{(n)}\|_2^2-\frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]^2}{b-a}\right)^{\frac{1}{2}}\\ =\ & \frac{1}{3}\frac{(b-a)^{n+\frac{1}{2}}}{2^n n!}\sqrt{\frac{2n^3-11n^2+18n-6}{4n^2-1}}\sqrt{\sigma(f^{(n)})}.\end{aligned}$$

We now suppose that (24) holds with a constant $C>0$ as

$$\begin{aligned}&\left|\int_a^b f(x)\,dx - \frac{b-a}{6}f(a|b) + \sum_{k=2}^{[\frac{n-1}{2}]}\frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}}f^{(2k)}\left(\frac{a+b}{2}\right)\right|\\ \leq\ & C(b-a)^{n+\frac{1}{2}}\sqrt{\sigma(f^{(n)})}.\end{aligned} \tag{25}$$

We may find a function $f:[a,b]\to\mathbf{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ as

$$f^{(n-1)}(x) = \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!} - \frac{(b-a)(x-a)^n}{6n!} & \text{if } x\in\left[a,\frac{a+b}{2}\right], \\ \frac{(x-b)^{n+1}}{(n+1)!} + \frac{(b-a)(x-b)^n}{6n!} & \text{if } x\in\left(\frac{a+b}{2},b\right]. \end{cases}$$

It follows that

$$f^{(n)}(x) = \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{6(n-1)!} & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{6(n-1)!} & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{26}$$

By (20)-(23) and (26), it is not difficult to find that the left-hand side of the inequality (25) becomes

$$L.H.S.(25) = \frac{(2n^3 - 11n^2 + 18n - 6)(b-a)^{2n+1}}{9(4n^2-1)(n!)^2 2^{2n}}, \tag{27}$$

and the right-hand side of the inequality (25) is

$$R.H.S.(25) = \frac{1}{3}\frac{1}{2^n n!}\sqrt{\frac{2n^3 - 11n^2 + 18n - 6}{4n^2-1}}C(b-a)^{2n+1}. \tag{28}$$

From (25), (27) and (28), we find that $C \geq \frac{1}{3}\frac{1}{2^n n!}\sqrt{\frac{2n^3-11n^2+18n-6}{4n^2-1}}$, proving that the constant $\frac{1}{3}\frac{1}{2^n n!}\sqrt{\frac{2n^3-11n^2+18n-6}{4n^2-1}}$ is the best possible in (24).

REMARK 2. It is clear that Theorem 1 and Theorem 4 can be regarded as special cases of Theorem 5.

THEOREM 6. Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ and $f^{(n)} \in L_2[a,b]$ where n is an even integer. Then we have

$$\begin{aligned} &\left| \int_a^b f(x)\,dx - \frac{b-a}{6}f(a|b) + \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right) \right. \\ &\left. + \frac{(n-2)(b-a)^n}{3(n+1)!2^n}[f^{(n-1)}(b) - f^{(n-1)}(a)] \right| \\ \leq\ &\frac{1}{3}\frac{(b-a)^{n+\frac{1}{2}}}{2^n(n+1)!}\sqrt{\frac{2n^5 - 11n^4 + 14n^3 + 4n^2 + 2n - 2}{4n^2-1}}\sqrt{\sigma(f^{(n)})}. \end{aligned} \tag{29}$$

Inequality (29) is sharp in the sense that the constant $\frac{1}{3}\frac{1}{2^n(n+1)!}\sqrt{\frac{2n^5-11n^4+14n^3+4n^2+2n-2}{4n^2-1}}$ cannot be replaced by a smaller one.

PROOF. From (20), (22), (23) and (2),we can easily get

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{b-a}{6} f(a|b) + \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right.\\
&\left.+\frac{(n-2)(b-a)^n}{3(n+1)!2^n}[f^{(n-1)}(b) - f^{(n-1)}(a)]\right|\\
= & \left|\int_a^b S_n(x) f^{(n)}(x)\,dx - \frac{1}{b-a}\int_a^b S_n(x)\,dx \int_a^b f^{(n)}(x)\,dx\right|\\
= & \frac{1}{2(b-a)}\left|\int_a^b\int_a^b [S_n(x)-S_n(t)][f^{(n)}(x)-f^{(n)}(t)]\,dx\,dt\right|\\
\le & \frac{1}{2(b-a)}\left\{\int_a^b\int_a^b [S_n(x)-S_n(t)]^2\,dx\,dt\right\}^{\frac12}\left\{\int_a^b\int_a^b [f^{(n)}(x)-f^{(n)}(t)]^2\,dx\,dt\right\}^{\frac12}\\
= & \left\{\int_a^b S_n^2(x)\,dx - \frac{1}{b-a}[\int_a^b S_n(x)\,dx]^2\right\}^{\frac12}\\
& \times\left\{\int_a^b [f^{(n)}(x)]^2\,dx - \frac{1}{b-a}[\int_a^b f^{(n)}(x)\,dx]^2\right\}^{\frac12}\\
= & \left\{\frac{(2n^5-11n^4+14n^3+4n^2+2n-2)(b-a)^{2n+1}}{9(4n^2-1)[(n+1)!]^2 2^{2n}}\right\}^{\frac12}\\
& \times\left\{\|f^{(n)}\|_2^2 - \frac{[f^{(n-1)}(b)-f^{(n-1)}(a)]^2}{b-a}\right\}^{\frac12}\\
= & \frac13\frac{(b-a)^{n+\frac12}}{2^n(n+1)!}\sqrt{\frac{2n^5-11n^4+14n^3+4n^2+2n-2}{4n^2-1}}\sqrt{\sigma(f^{(n)})}.
\end{aligned}$$

We now suppose that (29) holds with a constant $C > 0$ as

$$\begin{aligned}
&\left|\int_a^b f(x)\,dx - \frac{b-a}{6} f(a|b) + \sum_{k=2}^{[\frac{n-1}{2}]} \frac{(k-1)(b-a)^{2k+1}}{3(2k+1)!2^{2k-1}} f^{(2k)}\left(\frac{a+b}{2}\right)\right.\\
&\left.+\frac{(n-2)(b-a)^n}{3(n+1)!2^n}[f^{(n-1)}(b) - f^{(n-1)}(a)]\right|\\
\le\ & C(b-a)^{n+\frac12}\sqrt{\sigma(f^{(n)})}. \qquad (30)
\end{aligned}$$

We may find a function $f : [a,b] \to \mathbf{R}$ such that $f^{(n-1)}$ is absolutely continuous on $[a,b]$ as

$$f^{(n-1)}(x) = \begin{cases} \frac{(x-a)^{n+1}}{(n+1)!} - \frac{(b-a)(x-a)^n}{6n!} + \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^{n+1}} & \text{if } x \in [a, \frac{a+b}{2}],\\ \frac{(x-b)^{n+1}}{(n+1)!} + \frac{(b-a)(x-b)^n}{6n!} - \frac{(n-2)(b-a)^{n+1}}{3(n+1)!2^{n+1}} & \text{if } x \in (\frac{a+b}{2}, b]. \end{cases}$$

It follows that

$$f^{(n)}(x) = \begin{cases} \frac{(x-a)^n}{n!} - \frac{(b-a)(x-a)^{n-1}}{6(n-1)!} & \text{if } x \in \left[a, \frac{a+b}{2}\right], \\ \frac{(x-b)^n}{n!} + \frac{(b-a)(x-b)^{n-1}}{6(n-1)!} & \text{if } x \in \left(\frac{a+b}{2}, b\right]. \end{cases} \tag{31}$$

By (20)-(23) and (31), it is not difficult to find that the left-hand side of the inequality (30) becomes

$$L.H.S.(30) = \frac{(2n^5 - 11n^4 + 14n^3 + 4n^2 + 2n - 2)(b-a)^{2n+1}}{9(4n^2-1)[(n+1)!]^2 2^{2n}}, \tag{32}$$

and the right-hand side of the inequality (30) is

$$R.H.S.(30) = \frac{1}{3}\frac{1}{2^n(n+1)!}\sqrt{\frac{2n^5 - 11n^4 + 14n^3 + 4n^2 + 2n - 2}{4n^2-1}}C(b-a)^{2n+1}. \tag{33}$$

From (30), (32) and (33), we find that $C \geq \frac{1}{3}\frac{1}{2^n(n+1)!}\sqrt{\frac{2n^5-11n^4+14n^3+4n^2+2n-2}{4n^2-1}}$, proving that the constant $\frac{1}{3}\frac{1}{2^n(n+1)!}\sqrt{\frac{2n^5-11n^4+14n^3+4n^2+2n-2}{4n^2-1}}$ is the best possible in (29).

REMARK 3. It is clear that Theorem 3 can be regarded as a special case of Theorem 6.

REMARK 4. If we take $n = 4$ in Theorem 6, we get a sharp perturbed Simpson type inequality as

$$\left|\int_a^b f(t)\,dt - \frac{1}{b-a}f(a|b) + \frac{(b-a)^4}{2880}[f^{(3)}(b) - f^{(3)}(a)]\right| \leq \frac{1}{2880}\sqrt{\frac{11}{14}}(b-a)^{\frac{9}{2}}\sqrt{\sigma(f^{(4)})}. \tag{34}$$

Also, it should be noticed that inequality (34) has been appeared in [3] without a proof of its sharpness but with a misprint.

4 Applications in Numerical Integration

We restrict further considerations to the applications of Theorem 3 and Theorem 4.

THEOREM 7. Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a, b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ and let the assumptions of Theorem 3 hold. Then we have

$$\left|\int_a^b f(x)\,dx - \frac{h}{6}\sum_{i=0}^{n-1} f(x_i|x_{i+1})\right| \leq \frac{(b-a)^{\frac{5}{2}}}{12\sqrt{30}n^2}\sqrt{\sigma(f'')}. \tag{35}$$

PROOF. From (4) in Theorem 3 we obtain

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{6}f(x_i|x_{i+1})\right| \leq \frac{h^{\frac{5}{2}}}{12\sqrt{30}}\left\{\int_{x_i}^{x_{i+i}} [f''(t)]^2\,dt - \frac{1}{h}[f'(x_{i+1}) - f'(x_i)]^2\right\}^{\frac{1}{2}}. \tag{36}$$

By summing (36) over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$\left|\int_a^b f(t)\,dt - \frac{h}{6}\sum_{i=0}^{n-1} f(x_i|x_{i+1})\right| \le \frac{h^{\frac{5}{2}}}{12\sqrt{30}}\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+i}}[f''(t)]^2\,dt - \frac{1}{h}[f'(x_{i+1})-f'(x_i)]^2\right\}^{\frac{1}{2}}. \tag{37}$$

By using the Cauchy inequality twice, it is not difficult to obtain

$$\begin{aligned}
&\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+1}}[f''(t)]^2\,dt - \frac{1}{h}[f'(x_{i+1})-f'(x_i)]^2\right\}^{\frac{1}{2}} \\
\le\ & \sqrt{n}\left\{\int_a^b [f''(t)]^2\,dt - \frac{n}{b-a}\sum_{i=0}^{n-1}[f'(x_{i+1})-f'(x_i)]^2\right\}^{\frac{1}{2}} \\
\le\ & \sqrt{n}\left\{\|f''\|_2^2 - \frac{[f'(b)-f'(a)]^2}{b-a}\right\}^{\frac{1}{2}}.
\end{aligned} \tag{38}$$

Consequently, the inequality (35) follows from (37) and (38).

THEOREM 8. Let $\pi = \{x_0 = a < x_1 < \cdots < x_n = b\}$ be a given subdivision of the interval $[a,b]$ such that $h_i = x_{i+1} - x_i = h = \frac{b-a}{n}$ and let the assumptions of Theorem 4 hold. Then we have

$$\left|\int_a^b f(x)\,dx - \frac{h}{6}\sum_{i=0}^{n-1} f(x_i|x_{i+1})\right| \le \frac{(b-a)^{\frac{7}{2}}}{48\sqrt{105}n^3}\sqrt{\sigma(f''')}. \tag{39}$$

PROOF. From (12) in Theorem 4 we obtain

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{6}f(x_i|x_{i+1})\right| \le \frac{h^{\frac{7}{2}}}{48\sqrt{105}}\left\{\int_{x_i}^{x_{i+i}}[f'''(t)]^2\,dt - \frac{1}{h}[f''(x_{i+1})-f''(x_i)]^2\right\}^{\frac{1}{2}}. \tag{40}$$

By summing (40) over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$\begin{aligned}
&\left|\int_a^b f(t)\,dt - \frac{h}{6}\sum_{i=0}^{n-1} f(x_i|x_{i+1})\right| \\
\le\ & \frac{h^{\frac{7}{2}}}{48\sqrt{105}}\sum_{i=0}^{n-1}\left\{\int_{x_i}^{x_{i+i}}[f'''(t)]^2\,dt - \frac{1}{h}[f(x''_{i+1})-f''(x_i)]^2\right\}^{\frac{1}{2}}.
\end{aligned} \tag{41}$$

By using the Cauchy inequality twice, it is not difficult to obtain

$$\begin{aligned}
& \sum_{i=0}^{n-1} \left\{ \int_{x_i}^{x_{i+1}} [f'''(t)]^2 \, dt - \frac{1}{h} [f''(x_{i+1}) - f''(x_i)]^2 \right\}^{\frac{1}{2}} \\
\leq \quad & \sqrt{n} \left\{ \int_a^b [f'''(t)]^2 \, dt - \frac{n}{b-a} \sum_{i=0}^{n-1} [f''(x_{i+1}) - f''(x_i)]^2 \right\}^{\frac{1}{2}} \\
\leq \quad & \sqrt{n} \left\{ \|f'''\|_2^2 - \frac{[f''(b) - f''(a)]^2}{b-a} \right\}^{\frac{1}{2}} . \qquad (42)
\end{aligned}$$

Consequently, the inequality (39) follows from (41) and (42).

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