Applied Mathematics E-Notes, 9(2009), 197-204 © ISSN 1607-2510
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On The Nevanlinna Direction Of Quasi-Meromorphic Mapping Dealing With Multiple Values*

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Received 10 July 2008

Abstract

In this paper, by using Ahlfors' theory of covering surfaces, we prove that for quasi-meromorphic mapping f satisfying $\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty$, there exists at least one Nevanlinna direction dealing with multiple values.

1 Introduction

In 1997, the value distribution theory of meromorphic functions due to R. Nevanlinna (see [3,7] for standard references) was extended to the corresponding theory of quasi-meromorphic mappings by Sun and Yang [1,6]. The singular direction for $f(z)$ is one of the main objects studied in the theory of value distribution of quasi-meromorphic mappings. In [6], Sun and Yang obtained an existence theorem of the Borel direction by the filling disc theorem of quasi-meromorphic mappings. Later, several types of singular directions have been introduced in the literature. In 1999, Chen and Sun[1] defined Nevanlinna directions of quasi-meromorphic mappings on the complex plane and proved that there exists at least one Nevanlinna direction for quasi-meromorphic mappings of infinite order and it is also one Borel direction with respect to the type function. In 2004, Liu and Yang [4] studied the connections between the Julia direction and the Nevanlinna direction of quasi-meromorphic mappings by applying a fundamental inequality of an angular domain of quasi-meromorphic mappings.

In 2006, Li and Gu [5] proved that for a quasi-meromorphic mapping f satisfying $\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty$, there exists at least one Nevanlinna direction. However, it was not discussed whether there exists one Nevanlinna direction dealing with its multiple values. In this paper we investigate this problem. In the following, some definitions and notations are given, which can be found in [6].

DEFINITION 1. Let f be a complex and continuous functions in a region D. If for any rectangle $R=\{x+iy; a<x<b, c<y<d\}$ in D, $f(x+iy)$ is an absolutely continuous function of y for almost every $x\in(a,b)$, and $f(x+iy)$ is an absolutely

*Mathematics Subject Classifications: 30D60

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continuous function of x for almost every $y \in (c,d)$, then f is said to be absolutely continuous on lines in the region D. We also call that f is ACL in D.

DEFINITION 2. Let f be a homemorphism from D to D'. If (i) f is ACL in D, and (ii) there exists $K \geq 1$ such that $f(z) = u(x,y) + iv(x,y)$ satisfies $|f_z| + |f_{\bar{z}}| \leq K(|f_z| - |f_{\bar{z}}|)$ a. e. in D, then f is called an univalent K-quasiconformal mapping in D. If D' is a region on Riemann sphere V, then f is named an univalent K-quasi-meromorphic mapping in D.

DEFINITION 3. (see [6]) Let f be a complex and continuous function in the region D. For every point z_0 in D, if there is a neighborhood $U(\subset D)$ and a positive integer n depending on z_0, such that

$$F(z) = \begin{cases} (f(z))^{\frac{1}{n}}, & f(z_0) = \infty, \\ (f(z) - f(z_0))^{\frac{1}{n}} + f(z_0), & f(z_0) \neq \infty. \end{cases}$$

is an univalent K-quasi-meromorphic mapping, then f is named n-valent K-quasi-merom-orphic mapping at point z_0. If f is n-valent K-quasi-meromorphic at every point of D, then f is called a K-quasi-meromorphic mapping in D.

Let V be the Riemann sphere whose diameter is 1. For any complex number a and any positive real number r, let $n(r,a)$ be the number of zero points of $f(z) - a$ in disc $|z| < r$, counted according to their multiplicities, $\bar{n}^{l)}(r,a)$ be the number of distinct zeros of $f(z) - a$ with multiplicity $\leq l$ in disc $|z| < r$. Let F_r be the covering surface $f(z) = u(x,y) + iv(x,y)$ on sphere V and $S(r,f)$ be the average covering times of F_r to V,

$$S(r,f) = \frac{|F_r|}{|V|} = \frac{1}{\pi}\int_0^r \int_0^{2\pi} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1+|f|^2)^2} r d\varphi dr,$$

where $|F_r|$ and $|V|$ are the areas of F_r and V respectively,

$$T(r,f) = \int_0^r \frac{S(r,f)}{r} dr,$$

$$N(r,a) = \int_0^r \frac{n(t,a) - n(0,a)}{t} dt + n(0,a)\log r,$$

$$\bar{N}^{l)}(r,a) = \int_0^r \frac{\bar{n}^{l)}(t,a) - \bar{n}^{l)}(0,a)}{t} dt + \bar{n}^{l)}(0,a)\log r.$$

Let $\Omega(\varphi_1, \varphi_2) = \{z \in \mathbb{C} : \varphi_1 < \arg z < \varphi_2\}(0 \leq \varphi_1 < \varphi_2 \leq 2\pi)$, we denote

$$S(r,\varphi_1,\varphi_2;f) = \frac{|F_r|}{|V|} = \frac{1}{\pi}\int_0^r \int_{\varphi_1}^{\varphi_2} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1+|f|^2)^2} r d\varphi dr,$$

$$T(r,\varphi_1,\varphi_2;f) = \int_0^r \frac{S(r,\varphi_1,\varphi_2;f)}{r} dr,$$

when $\varphi_1 = 0, \varphi_2 = 2\pi$, we note $S(r,0,2\pi;f) = S(r,f), T(r,0,2\pi;f) = T(r,f)$.

For any complex number a, let $n(r,\varphi_1,\varphi_2;a)$ be the number of zero points of $f(z)-a$ in sector $\Omega(\varphi_1,\varphi_2)\cap\{z:|z|<r\}$, counted according to their multiplicities, $\bar{n}^{l)}(r,\varphi_1,\varphi_2;a)$ the number of distinct zeros of $f(z)-a$ with multiplicity $\le l$ in sector $\Omega(\varphi_1,\varphi_2)\cap\{z:|z|<r\}$. We define

$$N(r,\varphi_1,\varphi_2;a)=\int_0^r\frac{n(t,\varphi_1,\varphi_2;a)-n(0,\varphi_1,\varphi_2;a)}{t}dt+n(0,\varphi_1,\varphi_2;a)\log r,$$

$$\bar{N}^{l)}(r,\varphi_1,\varphi_2;a)=\int_0^r\frac{\bar{n}^{l)}(t,\varphi_1,\varphi_2;a)-\bar{n}^{l)}(0,\varphi_1,\varphi_2;a)}{t}dt+\bar{n}^{l)}(0,\varphi_1,\varphi_2;a)\log r.$$

Next we give the definitions concerning the Nevanlinna direction of quasi-meromorphic mappings dealing with multiple values.

DEFINITION 4. Let f be a K-quasi-meromorphic mapping and $l(\ge 3)$ be a positive integer. Then we call $\Theta^{l)}(a,\varphi_0)$ the deficiency of the value a in the direction $\Delta(\varphi_0)$: $\arg z=\varphi_0, 0\le\varphi_0<2\pi$. We call a the deficiency value of f in the direction $\Delta(\varphi_0)$ if $\Theta^{l)}(a,\varphi_0)>0$, where

$$\Theta^{l)}(a,\varphi_0)=1-\limsup_{\varepsilon\to+0}\limsup_{r\to\infty}\frac{\bar{N}^{l)}(r,\varphi_0-\varepsilon,\varphi_0+\varepsilon;a)}{T(r,\varphi_0-\varepsilon,\varphi_0+\varepsilon;f)}.$$

DEFINITION 5. We call $\Delta(\varphi_0):\arg z=\varphi_0$ the Nevanlinna direction of f dealing with multiple values if

$$\sum_{a\in\mathbb{C}\cup\{\infty\}}\Theta^{l)}(a,\varphi_0)\le\frac{2(l+1)}{l}$$

holds for any finitely many deficient value a, where $l(\ge 3)$ is a positive integer.

In this paper, we will prove the following theorem which improves the corresponding result in [5].

THEOREM 1. Let f be the K-quasi-meromorphic mapping and $l(\ge 3)$ be a positive integer. If

$$\limsup_{r\to\infty}\frac{T(r,f)}{(\log r)^2}=+\infty,$$

then there exists at least one Nevanlinna direction dealing with multiple values.

2 Some Lemmas

Let F be a finite covering surface of F_1, F is bounded by a finite number of analytic closed Jordan curves, its boundary is denoted by ∂F. We call the part of ∂F, which lies the interior of F_1, the relative boundary of F, and denote its length by L. Let D be a domain of F_1, its boundary consists of finite number of points or analytic closed Jordan curves, and $F(D)$ be the part of F, which lies above D. We denote the area of $F, F_1, F(D)$ and D by $|F|, |F_1|, |F(D)|$ and $|D|$, respectively. We call

$$S=\frac{|F|}{|F_1|},\qquad S(D)=\frac{|F(D)|}{|D|}$$

the mean covering numbering of F relative to F_1, D, respectively.

LEMMA 1. (See [2, Lemma 4]) Let F be a simply connected finite covering surfaces on the unit sphere V, $D_j (j = 1, 2, \ldots, q)$ be $q(\geq 3)$ disjoint disks with radius $\delta(> 0)$, and $n_j^{l)}$ be the number of simply connected islands in $F(D_j)$, which consist of not more than l sheets, then

$$\sum_{\nu=1}^{q} n_{\nu}^{l)} > (q - 2 - \frac{2}{l})S - \frac{C}{\delta^3}L,$$

where $C = 960 + 2\pi q$ and $l \geq 3$ is a positive integer.

LEMMA 2. (See [5, Lemma 2.2] Let $f(z)$ be a K-quasi-meromorphic mapping on the angular domain $\Omega(\varphi_0 - \delta, \varphi_0 + \delta)$, $a_1, \ldots, a_q (q \leq 3)$ are distinct points on the unit sphere V and the spherical distance of any two points is no small than $\gamma \in (0, \frac{1}{2})$. Let $F_0 = V \setminus \{a_1, a_2, \ldots, a_q\}$, $D = \Omega(r, \varphi_0 - \varphi, \varphi_0 + \varphi) \cap \{z : |z| > 1\} \setminus \{f^{-1}(a_1), f^{-1}(a_2), \ldots, f^{-1}(a_1)\}$ and $D_r = D \cap \{z : |z| < r\} (r > 1)$, $F_r = f(D_r) \subset V$. Then for any positive number φ satisfying $0 < \varphi < \delta$, we have

$$\begin{aligned} L(\partial f(D_r)) \leq \quad & \sqrt{2K}\pi \left[\tfrac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\right]^{\frac{1}{2}} (\log r)^{\frac{1}{2}} \\ & +\sqrt{2K\delta r}\mu^{\frac{1}{2}}(r, \varphi_0 - \delta, \varphi_0 + \delta) + \sqrt{2K\delta}\mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta), \end{aligned} \tag{1}$$

where F_r is the covering surface of F_0 and $L(\partial f(D_r))$ is the length of the relative boundary of F_r relative to F_0, and

$$\mu(r, \varphi_0 - \delta, \varphi_0 + \delta) = \int_{\varphi_0-\delta}^{\varphi_0+\delta} \frac{|f_z|^2 - |f_{\bar{z}}|^2}{(1 + |f(re^{i\varphi}|^2)^2} r d\varphi.$$

LEMMA 3. Let $f(z)$ be a K-quasi-meromorphic mapping on the angular domain $\Omega(\varphi_0 - \delta, \varphi_0 + \delta)$, $a_1, \ldots, a_q (q \leq 3)$ are distinct points on the unit sphere V and the spherical distance of any two points is no small than $\gamma \in (0, \frac{1}{2})$. Then

$$\begin{aligned} & (q - 2 - \tfrac{2}{l})S(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ \leq \quad & \textstyle\sum_{j=1}^{q} \bar{n}^{l)}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + \frac{2C^2\gamma^{-6}\pi^2 K}{(q-2-\frac{2}{l})(\delta-\varphi)} \log r \\ & +(q - 2 - \tfrac{2}{l})S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ & +2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}r^{\frac{1}{2}}\mu^{\frac{1}{2}}(r, \varphi_0 - \delta, \varphi_0 + \delta) \\ & +2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta) \end{aligned} \tag{2}$$

and

$$\begin{aligned} & (q - 2 - \tfrac{2}{l})T(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ \leq \quad & \textstyle\sum_{j=1}^{q} \bar{N}^{l)}(r, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + \frac{2C^2\gamma^{-6}\pi^2 K}{(q-2-\frac{2}{l})(\delta-\varphi)} (\log r)^2 \\ & +(q - 2 - \tfrac{2}{l})T(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \\ & +(q - 2 - \tfrac{2}{l})S(1, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \log r \\ & +2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\mu^{\frac{1}{2}}(1, \varphi_0 - \delta, \varphi_0 + \delta) \log r + \lambda(r, \varphi_0 - \delta, \varphi_0 + \delta) \end{aligned} \tag{3}$$

for any $\varphi, 0 < \varphi < \delta$, where C is a constant depending only on $\{a_1, a_2, \ldots, a_q\}$. $\lambda(r, \varphi_0 - \delta, \varphi_0 + \delta) = 2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}} \int_1^r (\frac{\mu(r,\varphi_0-\delta,\varphi_0+\delta)}{r})^{\frac{1}{2}} dr$,

$$\lambda(r, \varphi_0 - \delta, \varphi_0 + \delta) \leq 2C\gamma^{-3}\delta^{\frac{1}{2}}\pi^{\frac{1}{2}}K^{\frac{1}{2}}(T(r, \varphi_0 - \delta, \varphi_0 + \delta; f))^{\frac{1}{2}} \log T(r, \varphi_0 - \delta, \varphi_0 + \delta; f) \tag{4}$$

outside a set E_δ of r at most, where E_δ consists of a series of intervals and satisfies $\int_{E_\delta}(r\log r)^{-1}dr<+\infty$.

PROOF. Under the condition of Lemma 3 and Lemma 2, we have

$$S(D_r)=S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f). \tag{5}$$

Using Lemma 1, we easily obtain

$$\begin{aligned}&(q-2-\tfrac{2}{l})[S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)]\\ &\le \textstyle\sum_{j=1}^{q}\bar{n}^{l)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)+C\gamma^{-3}L(\partial(D_r)).\end{aligned} \tag{6}$$

where C is a constant depending only on $\{a_1,a_2,\dots,a_q\}$.

Taking (1) into (6), we have

$$\begin{aligned}&(q-2-\tfrac{2}{l})[S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)]\\ &-\textstyle\sum_{j=1}^{q}\bar{n}^{l)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)-C\gamma^{-3}\sqrt{2K\delta r}\mu^{\frac12}(r,\varphi_0-\delta,\varphi_0+\delta)\\ &-C\gamma^{-3}\sqrt{2K\delta}\mu^{\frac12}(1,\varphi_0-\delta,\varphi_0+\delta)\\ &\le C\gamma^{-3}\sqrt{2K}\pi\left[\tfrac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\right]^{\frac12}(\log r)^{\frac12}.\end{aligned} \tag{7}$$

We denote

$$\begin{aligned}A(r,\varphi)=\ &(q-2-\tfrac{2}{l})[S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)]\\ &-\textstyle\sum_{j=1}^{q}\bar{n}^{l)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j)-C\gamma^{-3}\sqrt{2K\delta r}\mu^{\frac12}(r,\varphi_0-\delta,\varphi_0+\delta)\\ &-C\gamma^{-3}\sqrt{2K\delta}\mu^{\frac12}(1,\varphi_0-\delta,\varphi_0+\delta).\end{aligned} \tag{8}$$

By (7) and (8), we have

$$A(r,\varphi)\le C\gamma^{-3}\sqrt{2K}\pi\left[\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\right]^{\frac12}(\log r)^{\frac12}. \tag{9}$$

From (8) we verify that $A(r,\varphi)$ is an increasing function of φ. Thus, there exists $\delta_0>0$, such that $A(r,\varphi)\le 0$ for $0<\varphi\le\delta_0$ and $A(r,\varphi)>0$ for $\varphi>\delta_0$.

We shall take the two cases in the following into consideration:

Case 1. For $\varphi>\delta_0$, by (9) we have

$$[A(r,\varphi)]^2\le 2C^2\gamma^{-6}K\pi^2\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}\log r. \tag{10}$$

By (8) we have

$$\frac{dA(r,\varphi)}{d\varphi}=\left(q-2-\frac{2}{l}\right)\frac{d(S(r,\varphi_0-\varphi,\varphi_0+\varphi;f)-S(1,\varphi_0-\varphi,\varphi_0+\varphi;f))}{d\varphi}. \tag{11}$$

From (10) and (11) we have

$$[A(r,\varphi)]^2\le\frac{2C^2\gamma^{-6}K\pi^2\log r}{q-2-\frac{2}{l}}\cdot\frac{dA(r,\varphi)}{d\varphi},$$

i.e.,

$$d\varphi \le \frac{2C^2\gamma^{-6}K\pi^2\log r}{q-2-\frac{2}{l}} \cdot \frac{dA(r,\varphi)}{[A(r,\varphi)]^2}.$$

Integrating two sides of the inequality leads to

$$\delta - \varphi = \int_{\varphi}^{\delta} d\varphi \le \frac{2C^2\gamma^{-6}K\pi^2\log r}{q-2-\frac{2}{l}} \int_{\varphi}^{\delta} \frac{dA(r,\varphi)}{[A(r,\varphi)]^2} \le \frac{2C^2\gamma^{-6}K\pi^2\log r}{q-2-\frac{2}{l}} \cdot \frac{1}{A(r,\varphi)}.$$

Thus

$$A(r,\varphi) \le \frac{2C^2\gamma^{-6}K\pi^2\log r}{\left(q-2-\frac{2}{l}\right)(\delta-\varphi)}. \tag{12}$$

Case 2. Because $A(r,\varphi) \le 0$ when $0 < \varphi \le \delta_0$, the above inequality also holds.

By Cases 1 and Case 2, we can easily get

$$A(r,\varphi) \le \frac{2C^2\gamma^{-6}K\pi^2\log r}{\left(q-2-\frac{2}{l}\right)(\delta-\varphi)},$$

for any $\varphi, 0 < \varphi < \delta$. Combining with the definition of $A(r,\varphi)$, we can easily get (2).

By dividing r and then integrating from 1 to r on two sides of (2) we get

$$\begin{array}{ll} & (q-2-\frac{2}{l})T(r,\varphi_0-\varphi,\varphi_0+\varphi;f) \\ \le & \sum_{j=1}^{q} \bar{N}^{l)}(r,\varphi_0-\delta,\varphi_0+\delta;a_j) + \frac{2C^2\gamma^{-6}\pi^2K}{(q-2-\frac{2}{l})(\delta-\varphi)}(\log r)^2 \\ & +(q-2-\frac{2}{l})T(1,\varphi_0-\varphi,\varphi_0+\varphi;f) \\ & +(q-2-\frac{2}{l})S(1,\varphi_0-\varphi,\varphi_0+\varphi;f)\log r \\ & +2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\mu^{\frac{1}{2}}(1,\varphi_0-\delta,\varphi_0+\delta)\log r \\ & +2C\gamma^{-3}\delta^{\frac{1}{2}}K^{\frac{1}{2}}\int_1^r[\frac{\mu(r,\varphi_0-\delta,\varphi_0+\delta)}{r}]^{\frac{1}{2}}dr. \end{array} \tag{13}$$

From the definitions of $S(r,\varphi_1,\varphi_2;f)$, $\mu(r,\varphi_0-\delta,\varphi_0+\delta)$ and $\lambda(r,\varphi_0-\delta,\varphi_0+\delta)$, and Schwarz's inequality we get

$$\begin{array}{ll} (\lambda(r,\varphi_0-\delta,\varphi_0+\delta))^2 & = 4C^2\gamma^{-6}\delta K\left[\int_1^r\left(\frac{\mu(r,\varphi_0-\delta,\varphi_0+\delta)}{r}\right)^{\frac{1}{2}}dr\right]^2 \\ & \le 4C^2\gamma^{-6}\delta K\int_1^r\mu(r,\varphi_0-\delta,\varphi_0+\delta)dr\int_1^r r^{-1}dr \\ & \le 4C^2\gamma^{-6}\pi\delta K\log r\int_1^r dS(r,\varphi_0-\delta,\varphi_0+\delta;f) \\ & \le 4C^2\gamma^{-6}\pi\delta KS(r,\varphi_0-\delta,\varphi_0+\delta;f)\log r \\ & = 4C^2\gamma^{-6}\pi\delta K\frac{dT(r,\varphi_0-\delta,\varphi_0+\delta;f)}{dr}r\log r. \end{array} \tag{14}$$

Choosing $r_0, r_0 > 0$ such that $T(r_0,\varphi_0-\delta,\varphi_0+\delta;f) > 1$, and setting $E_\delta = \{r_0 < r < \infty : (\lambda(r,\varphi_0-\delta,\varphi_0+\delta))^2 > 4C^2\gamma^{-6}\pi\delta KT(r,\varphi_0-\delta,\varphi_0+\delta;f)(\log T(r,\varphi_0-\delta,\varphi_0+\delta;f))^2\}$, we have

$$\begin{array}{ll} \int_{E_\delta}\frac{dr}{r\log r} & \le \int_{E_\delta}\frac{dT(r,\varphi_0-\delta,\varphi_0+\delta;f)}{T(r,\varphi_0-\delta,\varphi_0+\delta;f)(\log T(r,\varphi_0-\delta,\varphi_0+\delta;f))^2} \\ & \le [\log T(r_0,\varphi_0-\delta,\varphi_0+\delta;f)]^{-1} < +\infty. \end{array} \tag{15}$$

Then when $r > r_0$ and $r \notin E_\delta$, (4) holds.

Thus, we complete the proof of Lemma 3.

3 Proof of Theorem 1

By hypothesis of Theorem 1, there exists an increasing sequence $r_n(r_n \to +\infty$, as $n \to \infty)$ such that

$$\lim_{n\to\infty} \frac{T(r_n, f)}{(\log r_n)^2} = +\infty.$$

By the finite covering theorem at $[0, 2\pi]$, we know that there exists some φ_0 such that $\varphi_0 \in [0, 2\pi]$ and

$$\limsup_{n\to\infty} \frac{T(r_n, \varphi_0 - \varphi, \varphi_0 + \varphi; f)}{T(r_n, f)} > 0 \tag{16}$$

for arbitrary $\varphi, 0 < \varphi < \varphi_0$. Hence, we claim that the direction $\Delta(\varphi_0) : \arg z = \varphi_0$ is the Nevanlinna direction in Theorem 1.

Otherwise, for an arbitrary positive number $\varepsilon_0 > 0$, there exists some complex numbers $\{a_j\}(j = 1, 2, \ldots, q)(q \geq 3)$ such that the following inequality holds:

$$\frac{l}{l+1} \sum_{j=1}^{q} \Theta^{l)}(a_j, \varphi_0) > 2 + 2\varepsilon_0.$$

From Definition 4, we have

$$\limsup_{\varphi\to+0} \limsup_{r\to+\infty} \sum_{j=1}^{q} \frac{\overline{N}^{l)}(r, \varphi_0 - \varphi, \varphi_0 + \varphi; a_j)}{T(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f)} < q - \frac{2(l+1)}{l} - \frac{2(l+1)\varepsilon_0}{l}.$$

Therefore, there exists some φ' such that $\varphi' > 0$ and the following inequality holds:

$$\limsup_{r\to+\infty} \sum_{j=1}^{q} \frac{\overline{N}^{l)}(r, \varphi_0 - \varphi, \varphi_0 + \varphi; a_j)}{T(r, \varphi_0 - \varphi, \varphi_0 + \varphi; f)} < q - \frac{2(l+1)}{l} - \frac{2(l+1)\varepsilon_0}{l} \tag{17}$$

for an arbitrary $\varphi, 0 < \varphi < \varphi'$.

For any $\varphi, 0 < \varphi < \varphi'$, we define an increasing function as follows:

$$T(\varphi) = \limsup_{n\to+\infty} \frac{T(r_n, \varphi_0 - \varphi, \varphi_0 + \varphi; f)}{T(r_n, f)}.$$

From (16) we deduce $T(\varphi) > 0$. So we have $0 < T(\varphi) \leq 1$. By the increasing of $T(\varphi)$ in the interval $[0, \varphi']$ and the continuous theorem for monotonous functions, we see that all discontinuous points of $T(\varphi)$ constitute a countable set at most. Then, by Lemma 3, we can get

$$(q - 2 - \frac{2}{l})T(r_n, \varphi_0 - \varphi, \varphi_0 + \varphi; f) \leq \sum_{j=1}^{q} \bar{N}^{l)}(r_n, \varphi_0 - \delta, \varphi_0 + \delta; a_j) + O(\log r_n)^2$$

$$+O((T(r_n, \varphi_0 - \delta, \varphi_0 + \delta; f))^{\frac{1}{2}} \log T(r_n, \varphi_0 - \delta, \varphi_0 + \delta; f)) \tag{18}$$

for $0 < \varphi < \delta < \varphi'$ and $r_n \notin E_\delta$.

From (17), (18) and the definition of $T(\varphi)$ we can obtain

$$\left(q - \frac{2(l+1)}{l}\right) T(\varphi) < \left(q - \frac{2(l+1)}{l} - \frac{2(l+1)\varepsilon_0}{l}\right) T(\delta). \tag{19}$$

From (16) we get

$$T(\varphi) \to T(\delta), \qquad \varphi \to \delta. \tag{20}$$

Combining (20) with (19), we can obtain $T(\delta) = 0$. This contradicts $T(\delta) > 0$. The proof is complete.

Acknowledgements. We thank the referee(s) for reading the manuscript very carefully and making a number of valuable and kind comments which improved the presentation. The research was supported by the Research Projects of Jingdezhen Ceramic Institute and the Soft Scientific and Technological Research Projects of Jiangxi Province ([2008]147).

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