Applied Mathematics E-Notes, 9(2009), 192-196 © ISSN 1607-2510
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A Remark On A Uniqueness Result For A Boundary Value Problem Of Eighth-Order*

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Received 21 June 2008

Abstract

A maximum principle is presented for a function defined on solutions of a class of eighth-order elliptic equations. As an application, the uniqueness of the solution for the corresponding boundary value problem in a strictly convex plane domain is established.

1 Introduction

Dunninger [2] developed a maximum principle from which follows the uniqueness for the classical solution of the boundary value problem

$$\begin{cases} \Delta^2 u + cu = f & \text{in } \Omega \subset \mathbb{R}^n, \\ u = g, \Delta u = h & \text{on } \partial\Omega, \end{cases}$$

where $c > 0$ is a constant.

We note that a uniqueness result for solutions of a more general fourth-order elliptic equation, under the same boundary conditions follows from Corollary 1 of [7].

The uniqueness question for solutions of the boundary value problem (here $a, b \geq 0$ and $c > 0$ in Ω)

$$\begin{cases} \Delta^3 u - a(x)\Delta^2 u + b(x)\Delta u - c(x)u = f & \text{in } \Omega \subset \mathbb{R}^n, \\ u = g,\ \Delta u = h,\ \Delta^2 u = i & \text{on } \partial\Omega, \end{cases}$$

has been settled in a satisfactory way by Schaefer [5] (the constant coefficient case with n=2) and Goyal and Goyal [3] (the constant and variable coefficient case).

In this note we consider classical solutions (i.e., $C^8(\Omega) \cap C^6(\overline{\Omega})$) of

$$\Delta^4 u - a(x)\Delta^3 u + b(x)\Delta^2 u - c(x)\Delta u + du = 0, \tag{1}$$

in the bounded plane domain Ω, where a, b, c and d satisfy (2)–(5), and present ([1]) a maximum principle for a certain function defined on the solutions of (1). Then we use

*Mathematics Subject Classifications: 35G15, 35J40.

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the maximum principle to prove a uniqueness result for the corresponding boundary value problem.

Throughout we shall be concerned with functions defined on a bounded domain $\Omega \subset \mathbb{R}^2$. We shall let ∇, Δ and Δ^m denote respectively the gradient operator, the Laplace operator, the m - times iterated Laplace operator.

2 Uniqueness Result

The uniqueness result can be inferred from the following maximum principle [1]

LEMMA 1. Let u be a classical solution of (1). Assume that

$$a > 0, \quad \Delta(1/a) \leq 0 \quad \text{in } \Omega, \tag{2}$$

$$b \geq 0 \quad \text{in } \Omega, \tag{3}$$

$$c > 0, \quad \Delta(1/c) \leq 0 \quad \text{in } \Omega, \tag{4}$$

and

$$d > 0 \tag{5}$$

are satisfied. Then the functional

$$\mathrm{P} = \frac{c(x)}{2}(\Delta u)^2 + \frac{a(x)}{2}\left(\Delta^2 u\right)^2 + d(|\nabla u|^2 - u\Delta u) + |\nabla(\Delta^2 u)|^2 - \Delta^2 u \Delta^3 u \tag{6}$$

assumes its maximum value on $\partial\Omega$. The result also holds if a and c are nonnegative constants.

THEOREM 1. There is at most one classical solution of the boundary value problem

$$\left\{ \begin{array}{ll} \Delta^4 u - a\Delta^3 u + b(x)\Delta^2 u - c\Delta u + du = f & \text{in } \Omega, \\ u = g, \quad \Delta u = h, \quad \Delta^2 u = i, \quad \Delta^3 u = j & \text{on } \partial\Omega, \end{array} \right. \tag{7}$$

where $a, c \geq 0$, b satisfies (3), d satisfies (5), and the curvature k of $\partial\Omega$ (Ω is a smooth domain) is strictly positive.

PROOF. The proof is similar to the proof of Theorem 2.3 in [1]. It is displayed here for completeness.

We suppose that u_1 and u_2 are two solutions of (7). Defining $v = u_1 - u_2$, we see that v satisfies (1) and

$$v = \Delta v = \Delta^2 v = \Delta^3 v = 0 \quad \text{on } \partial\Omega. \tag{8}$$

By virtue of Lemma 1

$$\mathrm{P} \leq \max_{\partial\Omega} \mathrm{P} \quad \text{in } \Omega. \tag{9}$$

Since $v = \Delta^2 v = 0$ on $\partial\Omega$, we have

$$|\nabla v| = \left|\frac{\partial v}{\partial n}\right| \quad \text{on } \partial\Omega \tag{10}$$

and

$$|\nabla(\Delta^2 v)| = \left|\frac{\partial(\Delta^2 v)}{\partial n}\right| \quad \text{on } \partial\Omega, \tag{11}$$

where $\partial/\partial n$ denotes the outward directed normal derivative operator.

Now suppose that

$$\frac{\partial v}{\partial n} = \frac{\partial(\Delta^2 v)}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{12}$$

By (8), (9), (10), (11) and (12) we get

$$\mathrm{P} \leq 0 \quad \text{in } \Omega,$$

which gives

$$-dv\Delta v - \Delta^2 v\Delta^3 v \leq 0 \quad \text{in } \Omega. \tag{13}$$

Integrating (13) over Ω and using Green's identity we obtain

$$\int_\Omega d|\nabla v|^2 + \int_\Omega |\nabla(\Delta^2 v)|^2 \leq 0.$$

Hence $v \equiv 0$ in $\overline{\Omega}$ by continuity.

Thus we will have the required uniqueness if we prove (12).

We now prove (12).

A calculation gives (using (10) and (11))

$$\frac{\partial \mathrm{P}}{\partial n} = c\Delta v\frac{\partial(\Delta v)}{\partial n} + a\Delta^2 v\frac{\partial(\Delta^2 v)}{\partial n} + 2d\frac{\partial v}{\partial n}\frac{\partial^2 v}{\partial n^2} - \\ d\Delta v\frac{\partial v}{\partial n} - dv\frac{\partial(\Delta v)}{\partial n} + 2\frac{\partial(\Delta^2 v)}{\partial n}\frac{\partial^2(\Delta^2 v)}{\partial n^2} - \Delta^3 v\frac{\partial(\Delta^2 v)}{\partial n} - \Delta^2 v\frac{\partial(\Delta^3 v)}{\partial n} \quad \text{on } \partial\Omega.$$

Using (8) we obtain

$$\frac{\partial \mathrm{P}}{\partial n} = 2d\frac{\partial v}{\partial n}\frac{\partial^2 v}{\partial n^2} + 2\frac{\partial(\Delta^2 v)}{\partial n}\frac{\partial^2(\Delta^2 v)}{\partial n^2} \quad \text{on } \partial\Omega. \tag{14}$$

By introducing normal coordinates in the neighbourhood of the boundary, we can write

$$\Delta v = \frac{\partial^2 v}{\partial n^2} + \frac{\partial^2 v}{\partial s^2} + k\frac{\partial v}{\partial n}, \tag{15}$$

where $\partial/\partial s$ denotes the tangential derivative operator.

Since $v = \Delta v = 0$ on $\partial\Omega$, relation (15) becomes

$$\frac{\partial^2 v}{\partial n^2} = -k\frac{\partial v}{\partial n}.$$

Similarly, since $\Delta^2 v = \Delta^3 v = 0$ on $\partial\Omega$ we have

$$\frac{\partial^2(\Delta^2 v)}{\partial n^2} = -k\frac{\partial(\Delta^2 v)}{\partial n}.$$

Hence (14) becomes

$$\frac{\partial \mathrm{P}}{\partial n} = -2dk\left(\frac{\partial v}{\partial n}\right)^2 - 2k\left(\frac{\partial(\Delta^2 v)}{\partial n}\right)^2 \leq 0 \quad \text{on } \partial\Omega.$$

This contradicts Hopf's lemma at a point $x_0 \in \partial\Omega$, where P (P $\not\equiv$ constant) assumes its maximum value (by Lemma 1).

Hence P constant in $\overline{\Omega}$.

Thus

$$\frac{\partial \mathrm{P}}{\partial n} = 0 \quad \text{on } \partial\Omega$$

and consequently (12) is established.

It is known that once we have a maximum principle for an equation, the nonexistence of a nontrivial solution of the zero - boundary problem will be a consequence. An inverse result, of establishing a maximum principle from some nonexistence results was carried out by Schaefer and Walter (Theorem 2, [6]).

Using their result and our Theorem 1, we obtain the following maximum principle

COROLLARY 1. Suppose that u is a classical solution of the boundary value problem

$$\begin{cases} \Delta^4 u - a\Delta^3 u + b\Delta^2 u - c\Delta u + du = 0 & \text{in } \Omega, \\ \Delta u = 0, \quad \Delta^2 u = 0, \quad \Delta^3 u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a, b, c \geq 0$, d satisfies (5), and the curvature k of $\partial\Omega$ (Ω is a smooth domain) is strictly positive. Then there exists a constant $K > 0$ such that

$$\max_{\overline{\Omega}} |u| \leq K \max_{\partial\Omega} |u|.$$

3 Remarks

1. If $a = b = c = d = 0$ in Theorem 1, then the dimension and geometry conditions are redundant (see Theorem 8, [4]).

2. We note that some sign conditions on the coefficients a, b, c, d are needed in Theorem 1 (and perhaps some geometry conditions) since $u_1(x, y) \equiv 0$ and $u_2(x, y) = \sin x \sin y$ satisfy

$$\begin{cases} \Delta^4 u - 16u = 0 & \text{in } \Omega, \\ u = \Delta u = \Delta^2 u = \Delta^3 u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (0, 2\pi) \times (0, 2\pi)$.

3. Various other P - functions could be derived as well.
For example the functions

$$\begin{aligned}
\mathrm{P}_1 \quad = \quad & (\Delta^3 u - a\Delta^2 u - bu)^2/2 + (d\Delta^2 u + u)^2/2 + (c-b)(|\nabla(\Delta u)|^2 - \Delta u \Delta^2 u) \\
& (b-d^2)(\Delta^2 u)^2/2 + (ad+b^2)(|\nabla u|^2 - u\Delta u) + (ac - ab - d)(\Delta u)^2/2 \\
& +(bc - b^2 - 1)u^2/2,
\end{aligned}$$

where $c > b \geq d^2$, $c - b \geq \max\{d/a, 1/b\}$, $d, a > 0$;

$$\begin{aligned}
\mathrm{P}_2 \quad = \quad & (a\Delta^3 u + d\Delta u)^2/2 + (a^2\Delta^2 u + du)^2/2 + a^2(c+d)(|\nabla(\Delta u)|^2 - \Delta u \Delta^2 u) \\
& +a(ab - d - a^3)(\Delta^2 u)^2/2 + (abd - d^2 - a^2 d)(\Delta u)^2/2 + d^2(a-1)u^2/2,
\end{aligned}$$

where $a \geq 1, ab - d - a^3 \geq 0$, $c \geq 0$, $b, d > 0$,
take a maximum on $\partial\Omega$ (u is a solution of (1)).
Analogously as before we are led to some uniqueness results for problem (7), which are weaker than the result stated in Theorem 1.

4. It seems very likely that the following is true:
There exists at most one classical solution ($C^{2m}(\Omega) \cap C^{2m-2}(\overline{\Omega}), m \geq 5$) of

$$\begin{cases} \Delta^m u - a_{m-1}\Delta^{m-1}u + a_{m-2}\Delta^{m-2}u + \cdots + (-1)^m a_0 u = f & \text{in } \Omega, \\ u = g_1,\, \Delta u = g_2, \ldots,\, \Delta^{m-1}u = g_m & \text{on } \partial\Omega, \end{cases}$$

where the constants $a_i \geq 0$, $i = 1, \ldots, m-1$, $a_0 > 0$ and the curvature k of $\partial\Omega$ ($\Omega \subset \mathbb{R}^2$ is a smooth domain) is strictly positive. This is still an open question.

Acknowledgement. This work was supported by a Katholischer Akademischer Auslander-Dienst fellowship (KAAD). The author is grateful to the KAAD foundation for support.

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