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Blowup For Degenerate and Singular Parabolic Equation With Nonlocal Source*

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Abstract

This paper deals with the blow-up properties of the solution to the degenerate and singular parabolic equation with nonlocal source and homogeneous Dirichlet boundary conditions. The sufficient conditions for the solution exists globally or blows up in finite time are obtained. Furthermore, we consider the global blow-up and the asymptotic behavior of blow-up solution.

1 Introduction

In this paper, we consider the following degenerate and singular nonlinear reaction-diffusion equation with nonlocal source

$$\begin{cases} |x|^q u_t - \operatorname{div}(|x|^\alpha \nabla u) = \int_\Omega f(u(y,t))dy, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \overline{\Omega}, \end{cases} \tag{1.1}$$

where $\Omega \subset R^N (N \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$, $T > 0$, $0 \leq u_0 \in C^{2+\gamma}(\overline{\Omega})$ with $\gamma \in (0,1)$, $|q| + \alpha \neq 0$ with $\alpha \in (0,2)$, $f \in C^1$ is defined in $[0,+\infty)$ with $f(s) \geq 0$ for $s \geq 0$. Since $u_0 \geq 0$ and $\int_\Omega f(0)dy \geq 0$, we know that $\underline{u} = 0$ is a subsolution of problem (1.1), then $u(x,t) \geq 0$ for $(x,t) \in \Omega \times (0,T)$ by comparison for parabolic equation (see [4]). So the term $\int_\Omega f(u(y,t))dy$ in first equation of (1.1) is well defined.

Let $\Omega_t = \Omega \times (0,t]$. Since $|q| + \alpha \neq 0$, the coefficients of $u_t,\ u_{x_i},\ u_{x_i x_i}$ may tend to 0 or ∞ as x tends to 0 $(i = 1, ..., n)$, we can regard the equation as degenerate and singular.

Floater [6] and Chan et al. [3] investigated the blow-up properties of the following degenerate parabolic problem

$$\begin{cases} x^q u_t - u_{xx} = u^p, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.2}$$

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where $q > 0$ and $p > 1$. Under certain conditions on the initial data $u_0(x)$, Floater [6] proved that the solution $u(x,t)$ of (1.2) blows up at the boundary $x = 0$ for the case $1 < p \leq q+1$. For the case $p > q+1$, in [3] Chan and Liu continued to study problem (1.2). Under certain conditions, they proved that $x = 0$ is not a blow-up point and the blow-up set is a proper compact subset of $(0, a)$.

For the case $q = 0$, in [7], the author showed that the blow-up set is a proper compact subset of $(0, a)$.

The motivation for studying problem (1.2) comes from Ockendon's model (see [9]) for the flow in a channel of a fluid whose viscosity depends on temperature

$$xu_t = u_{xx} + e^u, \tag{1.3}$$

where u represents the temperature of the fluid. In [6] Floater approximated e^u by u^p and considered equation (1.2).

Budd et al. [2] generalized the results in [6] to the following degenerate quasilinear parabolic equation

$$x^q u_t = (u^m)_{xx} + u^p, \tag{1.4}$$

with homogeneous Dirichlet conditions in the critical exponent $q = \frac{p-1}{m}$, where $q > 0$, $m \geq 1$ and $p > 1$. They pointed out that the general classification of blow-up solution for the degenerate equation (1.4) stays the same for the quasilinear equation (see [2] and [10])

$$u_t = (u^m)_{xx} + u^p. \tag{1.5}$$

In [5], Chen et al. discussed the following degenerate and singular semilinear parabolic equation

$$\begin{cases} u_t - (x^\alpha u_x)_x = \int_0^a f(u(x,t))dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.6}$$

they established the local existence and uniqueness of classical solution. Under appropriate hypotheses, they obtained some sufficient conditions for the global existence and blow-up of positive solution.

In [4], Chen et al. consider the following degenerate nonlinear reaction-diffusion equation with nonlocal source

$$\begin{cases} x^q u_t - (x^\gamma u_x)_x = \int_0^a u^p dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a], \end{cases} \tag{1.7}$$

they established the local existence and uniqueness of classical solution. Under appropriate hypotheses, they also got some sufficient conditions for the global existence and blow-up of positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution is the whole domain.

In [1], Abdellaoui et al. study the following parabolic problem

$$\begin{cases} u_t - \operatorname{div}(|x|^{-p\gamma}|\nabla u|^{p-2}\nabla u) = \lambda \frac{u^\alpha}{|x|^{p(\gamma+1)}}, & u \geq 0,\ (x,t) \in \Omega \times (0,T), \\ u\chi_{\Sigma_1 \times (0,T)} + |x|^{-p\gamma}|\nabla u|^{p-2}\frac{\partial u}{\partial \nu}\chi_{\Sigma_2 \times (0,T)} = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = \varphi(x), & x \in \Omega, \end{cases} \tag{1.8}$$

where $\Omega \subset R^N$ is a smooth bounded domain with $0 \in \Omega$, $\alpha \geq p-1$ and $-\infty < \gamma < (N-p)/p$, $\Sigma_i \subset \partial\Omega$, $(i = 1, 2)$ are two smooth $(N-1)-$dimensional manifolds, $\Sigma_1 \cap \Sigma_2 = \varnothing$, $\overline{\Sigma_1} \cup \overline{\Sigma_2} = \partial\Omega$ and $\overline{\Sigma_1} \cap \overline{\Sigma_2}$ is the interface, which is a smooth $(N-2)-$dimensional manifold.

They give some existence, nonexistence and complete blow-up results related to some Hardy-Soblev inequalities and a weak version of Harnack inequality, that holds for $p \geq 2$ and $\gamma + 1 > 0$.

In this paper, we generalize the results of [4] to multi-dimension and investigate the effect of the singularity, degeneracy and nonlocal reaction on the behavior of the solution of (1.1). We consider (1.1) of a special case, that is $u(x,t)$ is radial in x, so we require that $u_0(x)$ is radial in x and $\Omega = B(0,1)$ is a unit ball in R^N $(N \geq 2)$.

Set $r = \sqrt{x_1^2 + x_2^2 + \cdots + x_N^2}$ and $u(x,t) = u(|x|,t) = u(r,t)$, then the equation (1.1) takes the following form

$$\begin{cases} r^q u_t - \left(r^\alpha u_{rr} + (N+\alpha-1) r^{\alpha-1} u_r\right) = \int_\Omega f(u(|x|,t))dx, & (x,t) \in \Omega \times (0,T), \\ u_r(0,t) = 0, \quad u(1,t) = 0, & t \in (0,T), \\ u(0,r) = u_0(r) = u_0(x), & 0 \leq r \leq 1. \end{cases} \tag{1.9}$$

Now we state our main results

Under the following assumption, we get the global-existence result

(H1)There exist a $(0 < a < +\infty)$, such that $a \geq \int_\Omega f(a\psi(|x|))dx$, where $\psi(r)$ is a solution of the following inequality

$$\begin{cases} -\left(r^\alpha \psi^{''}(r) + (N+\alpha-1) r^{\alpha-1} \psi^{'}(r)\right) \geq 1, \ x \in (0,1), \\ 0 < \psi(0) < +\infty, \ \psi^{'}(0) \leq 0, \ \psi(1) \geq 0, \end{cases} \tag{1.10}$$

and it is given by $\psi(r) = \frac{1}{N(2-\alpha)}\left((r+\varepsilon)^{2-N-\alpha} - r^{2-\alpha} + \varsigma\right)$ for any constants $\varepsilon > 0$, $\varsigma \geq 1$.

REMARK 1. We can choose a for a large range of $f(u(x,t))$. For example, if $f(u(x,t)) = u^p(x,t)$ $(p > 1)$, then we can choose

$$0 < a = \left(w_N \left(\frac{1}{N(2-\alpha)}\right)^p \int_0^1 r^{N-1} \left((r+\varepsilon)^{2-N-\alpha} - r^{2-\alpha} + \varsigma\right)^p dr\right)^{-1/(p-1)},$$

where w_N is the volume of the unit ball in R^N and any constants $\varepsilon > 0$ and $\varsigma \geq 1$.

THEOREM 1.1. Let (H1) holds and $u(r,t)$ be the solution of (1.9). If $u_0(r) \leq a\psi(r)$, then $u(r,t)$ exists globally.

The blow-up results relies on the following assumptions

(H2) $q > \alpha - 1$ and $q \geq 0$.

(H3) The nonnegative function $f(s)$ satisfies $f \in C([0,+\infty)) \cap C^1((0,+\infty))$, $f^{'}(s) > 0$ for $s > 0$. $f(s)$ is convex and for some $s_0 > 0$, $\int_{s_0}^{+\infty} \frac{ds}{f(s)} < +\infty$.

THEOREM 1.2. Let (H2)-(H3) hold, then the solution of (1.1) blows up in finite time if $u_0(x)$ is large enough.

REMARK 2. For $f(s) = s^p$, from Theorem 1.1 and 1.2, we get if $p > 1$ the solution of (1.9) blows up in finite time for large initial data, while global existence for small initial data; if $p < 1$, for any initial data, the solution of (1.1) is global existence.

In the last, we consider the global blow-up and asymptotic behavior for the special case $q = 0$ under the following assumption.

(H4) There exists some constant $M < +\infty$, such that $\operatorname{div}(|x|^\alpha \nabla u_0(x)) \le M$ in Ω.

THEOREM 1.3. If (H2)-(H4) hold, the solution of (1.1) blows up in finite time T^*, then we have
(i) If $f(u) = u^p (p > 1)$, then $\lim_{t\to T^*}(T^* - t)^{1/(p-1)} u(x,t) = ((p-1)|\Omega|)^{-1/(p-1)}$ on any compact subset $\Omega' \subset\subset \Omega$.
(ii) If $f(u) = e^u$, then $\lim_{t\to T^*} |\log(T^* - t)| u(x,t) = 1$ on any compact subset $\Omega' \subset\subset \Omega$.

REMARK 3. From (H2) and (H4), we know that $0 < \alpha < 1$ and we can choose a large of $u_0(x)$ to satisfy (H2)-(H4), i.e., $u_0(x) = |x|^{3-\alpha}$.

Since we consider the radial solution, the proofs of the local existence of classical solution and comparison principle are similar to [4]. This paper is organized as follows. In the next section, we give some criteria for the solution $u(x,t)$ to exists globally or blow-up in finite time. In the last we consider the global blow-up and the asymptotic behavior of the blow-up solution.

2 Global Existence and Blow-up of the Solution

In this section, we give the proof of Theorem 1.1.

PROOF of Theorem 1.1. Let $\overline{u} = a\psi(r)$, then we have

$$\begin{array}{ll} r^q \overline{u}_t(r,t) - \left(r^\alpha \overline{u}_{rr}(r,t) + (N+\alpha-1) r^{\alpha-1}\overline{u}_r(r,t)\right) & \\ = -a\left(r^\alpha \psi''(r) + (N+\alpha-1) r^{\alpha-1}\psi'(r)\right) & \\ \ge a \ge \int_\Omega f(a\psi(|x|))dx, & (r,t)\in(0,1)\times(0,T), \\ -\overline{u}_r(0,t) = -a\psi'(0) \ge 0,\ \overline{u}(1) = a\psi(1) \ge 0, & t\in(0,T), \\ \overline{u}(r,0) = a\psi(r) \ge u_0(r), & 0 \le r \le 1, \end{array}$$

that is to say $\overline{u}(r,t) = a\psi(r)$ is a supersolution of (1.9). The proof of Theorem 1.1 is complete.

Next, we give some blow-up result of the solution of (1.1) under the assumptions of (H2)-(H3). First, we consider the following eigenvalue problem

$$\begin{cases} -\left(r^\alpha \varphi''(r) + (N+\alpha-1) r^{\alpha-1}\varphi'(r)\right) = \lambda r^q \varphi(r), \quad r\in(0,1), \\ 0 < \varphi(0) < +\infty, \quad \varphi(1) = 0. \end{cases} \tag{2.1}$$

By transformation $\varphi(r) = r^{\frac{2-\alpha-N}{2}}\xi(r)$, the above differential equation becomes

$$\begin{cases} r^2\xi''(r) + r\xi'(r) - \frac{(N+\alpha-2)^2}{4}\xi(r) + \lambda r^{q+2-\alpha}\xi(r) = 0, \quad r\in(0,1), \\ \xi(0) = 0,\ \xi(1) = 0. \end{cases} \tag{2.2}$$

Again, by transformation $\xi(r) = \eta(s)$, $r = s^{\frac{2}{q+2-\alpha}}$, the problem (2.2) becomes

$$\begin{cases} s^2\eta''(s) + s\eta'(s) + \left(\frac{4\lambda s^2}{(q+2-\alpha)^2} - \frac{(N+\alpha-2)^2}{(q+2-\alpha)^2}\right)\eta(s) = 0, \quad s\in(0,1), \\ \eta(0) = 0, \quad \eta(1) = 0. \end{cases} \tag{2.3}$$

Equation (2.3) is a Bessel equation. Its general solution is given by

$$\eta(s) = AJ_{\frac{N+\alpha-2}{q-\alpha+2}}\left(\frac{2\sqrt{\lambda}}{q+2-\alpha}s\right) + BJ_{-\frac{N+\alpha-2}{q-\alpha+2}}\left(\frac{2\sqrt{\lambda}}{q+2-\alpha}s\right),$$

where A and B are arbitrary constants, $J_{\frac{N+\alpha-2}{q-\alpha+2}}$ and $J_{-\frac{N+\alpha-2}{q-\alpha+2}}$ denote Bessel functions of the first kind of orders $\frac{N+\alpha-2}{q-\alpha+2}$ and $-\frac{N+\alpha-2}{q-\alpha+2}$, respectively. Let μ be the first root of $J_{\frac{N+\alpha-2}{q-\alpha+2}}\left(\frac{2\sqrt{\lambda}}{q+2-\alpha}\right)$. By Mclachlan [8, pp. 29 and 75], it is positive. It is obvious that μ is the first eigenvalue of problem (2.1); also we can easily obtain the corresponding eigenfunction

$$\varphi(r) = kr^{\frac{2-\alpha-N}{2}} J_{\frac{N+\alpha-2}{q-\alpha+2}}\left(\frac{2\sqrt{\mu}}{q+2-\alpha} r^{\frac{q+2-\alpha}{2}}\right), \tag{2.4}$$

since $q > \alpha - 1$, we can choose k such that $\int_\Omega \varphi(|x|)dx = 1$.

PROOF of Theorem 1.2. We set $U(t) = \int_\Omega |x|^q \varphi(|x|)u(x,t)dx$, then from equation (1.1) and (2.1), we have

$$\begin{aligned} U^{'}(t) &= \int_\Omega |x|^q \varphi(|x|)u_t(x,t)dx = \int_\Omega \left(\operatorname{div}(|x|^\alpha \nabla u(x,t)) + \int_\Omega f(u(x,t))dx\right)\varphi(|x|)dx \\ &= -\mu \int_\Omega |x|^q u(x,t)\varphi(|x|)dx + \int_\Omega f(u(x,t))dx. \end{aligned} \tag{2.5}$$

Since $f(s)$ is convex and nondecreasing from (H3), $|x|^q \le 1$ from (H2). Using Jensen's inequality, we have

$$\int_\Omega f(u(x,t))dx \ge |\Omega| f\left(\frac{1}{|\Omega|}\int_\Omega u(x,t)dx\right) \ge |\Omega| f\left(\frac{1}{|\Omega|}\int_\Omega |x|^q u(x,t)dx\right). \tag{2.6}$$

Take $c_0 = \max_{x\in\overline{\Omega}} \varphi(|x|)$, then $c_0 > 0$ and

$$U(t) = \int_\Omega |x|^q \varphi(|x|)u(x,t)dx \le c_0 \int_\Omega |x|^q u(x,t)dx. \tag{2.7}$$

Since (H3), f is nondecreasing, then we have

$$f\left(\frac{1}{|\Omega|}\int_\Omega |x|^q u(x,t)dx\right) \ge f\left(\frac{1}{c_0|\Omega|}\int_\Omega |x|^q \varphi(|x|)u(x,t)dx\right). \tag{2.8}$$

Now from (2.5)-(2.8), we get the following inequality

$$U^{'}(t) \ge -\mu U(t) + |\Omega| f\left(\frac{1}{c_0|\Omega|}U(t)\right). \tag{2.9}$$

By the condition $\int_{s_0}^{+\infty} \frac{ds}{f(s)} < +\infty$ from assumption (H3), we claim

$$\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty. \tag{3.10}$$

In fact, from the condition $\int_{s_0}^{+\infty} \frac{ds}{f(s)} < +\infty$, we know that $\lim_{s\to+\infty} f(s) = +\infty$. Since f is convex, $f^{'}(s)$ is nondecreasing. By L'Hospital principle, we have

$$\lim_{s\to+\infty} \frac{f(s)}{s} = \lim_{s\to+\infty} f^{'}(s). \tag{2.11}$$

If the claim is not true, from (2.11), we may assume $\lim_{s\to+\infty} f^{'}(s) = M < +\infty$, then there exists $s_1 \geq s_0$ such that $f(s) \leq 3/2Ms$ for $s \geq s_1$, then

$$\int_{s_0}^{+\infty} \frac{ds}{f(s)} \geq \frac{2}{3M} \int_{s_1}^{+\infty} \frac{ds}{s} = +\infty. \tag{2.12}$$

(2.12) is contradict to the assumption (H3), so the claim (2.10) is true. Since $\mu > 0$, from (2.10), there exists $s_2 > s_0$, such that $f(s)/s \geq 2c_0\mu$ for $s \geq s_2$. So we have the following inequality

$$f(s) - \mu c_0 s \geq \frac{f(s)}{2}, \quad s \geq s_2. \tag{2.13}$$

Take $u_0(x)$ large enough such that

$$\int_\Omega |x|^q u_0(x)\varphi(|x|)dx \geq c_0|\Omega|s_2. \tag{2.14}$$

Using (2.13), (2.14) and integrating (2.9) from 0 to T, then we have

$$\begin{aligned} T &\leq \int_0^T \frac{dU(t)}{-\mu U(t) + |\Omega| f\left(U(t)/(c_0|\Omega|)\right)} = c_0 \int_0^T \frac{d(U(t)/(c_0|\Omega|))}{-\mu c_0(U(t)/(c_0|\Omega|)) + f\left(U(t)/(c_0|\Omega|)\right)} \\ &\leq c_0 \int_{U(0)/(c_0|\Omega|)}^{U(T)/(c_0|\Omega|)} \frac{2ds}{f(s)} \leq c_0 \int_{s_2}^{+\infty} \frac{2}{f(s)} \leq 2c_0 \int_{s_0}^{+\infty} \frac{ds}{f(s)} < +\infty, \end{aligned}$$

which means $u(x,t)$ blows up in a finite time. The proof of Theorem 3.1 is complete.

3 Global Blow-up and Asymptotic Behavior

In this section, we will prove if the solution of (1.1) blows up in finite T^*, then the blow-up set is the whole domain Ω under the assumption $q = 0$. We consider the asymptotic behavior of the blow-up solution in special case.

LEMMA 3.1. If (H2)-(H4) hold, the solution of (1.1) satisfies

$$\operatorname{div}(|x|^\alpha \nabla u(x,t)) \leq M, \quad (x,t) \in \Omega \times (0,T). \tag{3.1}$$

PROOF. Set $v(x,t) = \operatorname{div}(|x|^\alpha \nabla u(x,t)) - M$, then (1.1) implies $v(x,t)$ satisfies the following equation

$$v_t = \operatorname{div}(|x|^\alpha \nabla v(x,t)), \quad (x,t) \in \Omega \times (0,T), \tag{3.2}$$

since $v(x,0) = \operatorname{div}(|x|^\alpha \nabla u_0(x)) - M \leq 0,\ x \in \Omega$ and $v(x,t)|_{\partial\Omega} = -\int_\Omega f(u(x,t))dx - M < 0$, we know $v(x,t) \leq 0$ in $\Omega \times (0,T)$ from comparison principle.

Set

$$g(t)=\int_\Omega f(u(x,t))dx, \quad G(t)=\int_0^t g(s)ds. \tag{3.3}$$

LEMMA 3.2. If (H2)-(H4) hold, the solution of (1.1) blows up in finite time T^*, then we have

$$\lim_{t\to T^*} g(t)=+\infty, \quad \lim_{t\to T^*} G(t)=+\infty. \tag{3.4}$$

PROOF. Set $x_0\in\overline{\Omega}$ is a blow-up point, then there exists $\{(x_n,t_n)\}_{n=1}^{+\infty}$, $(x_n,t_n)\in\Omega\times(0,T^*)$ such that $(x_n,t_n)\to(x_0,T^*)$, $u(x_n,t_n)\to+\infty$ as $n\to+\infty$. For any $t\in(0,T^*)$, integrating (1.1) over $(0,t)$, then

$$u(x,t)=u_0(x)+\int_0^t \operatorname{div}(|x|^\alpha\nabla u(x,t))ds+G(t), \tag{3.5}$$

since $M_0=\max_{x\in\overline{\Omega}} u_0(x)<+\infty$, we get from (3.5) and Lemma 3.1 that

$$u(x,t)\le C_1+G(t), \quad (x,t)\in\Omega\times(0,T^*), \tag{3.6}$$

where $C_1=M_0+MT^*$, then $u(x_n,t_n)\le C+G(t_n)$. So $\lim_{n\to+\infty} G(t_n)=+\infty$, then by the nondecreasing property of $G(t)$ we get $\lim_{t\to T^*} G(t)=+\infty$. Since $T^*<+\infty$, it is easy to prove $\lim_{t\to T^*} g(t)=+\infty$.

Now we can prove the global blow-up result

LEMMA 3.3. If (H2)-(H4) hold, the solution of (1.1) blows up in finite time T^*, then we have

$$\lim_{t\to T^*}\frac{u(x,t)}{G(t)}=1. \tag{3.7}$$

PROOF of Lemma 3.3 and Theorem 1.3. First, we consider equation (1.9) and make the following transformation

$$v(r,t)=w(s,t), \quad r=((2-N-\alpha)s)^{1/(2-N-\alpha)}, \tag{4.8}$$

then equation (1.9) becomes

$$\begin{cases} w_t-d_0s^{-\beta}w_{ss}=g(t), & (s,t)\in(-\infty,l)\times(0,t),\\ w_s(-\infty,t)=0,\ w(l,t)=0, & t\in(0,T),\\ w(s,0)=w_0(s), & s\in(-\infty,l], \end{cases} \tag{3.9}$$

where $d_0=(2-N-\alpha)^{-\beta}$, $\beta=(2N+\alpha-2)/(2-N-\alpha)$, $l=1/(2-N-\alpha)$, $w_0(s)=u_0(((2-N-\alpha)s)^{1/(2-N-\alpha)})$. The remaining proof is similar to [5], so we omit it. The proof of Lemma 3.3 is complete.

From the above Lemma, we know that the blow-up set is the whole domain Ω. For the special case of $f(u(x,t))$, similar to the proof of Theorem 2.1 of [11], we can prove Theorem 1.3.

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