Applied Mathematics E-Notes, 9(2009), 177-183 © ISSN 1607-2510
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A Concise Proof For Properties Of Three Functions Involving The Exponential Function*

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Received 26 May 2008

Abstract

In this note a concise proof is supplied for some properties of three elementary functions involving the exponential function and relating to the remainder of Binet's first formula for the logarithm of the gamma function.

1 Introduction

Let us define three functions:

$$f(t) = \begin{cases} \dfrac{1}{t^2} - \dfrac{e^{-t}}{(1-e^{-t})^2}, & t \neq 0; \\ \dfrac{1}{12}, & t = 0; \end{cases} \tag{1}$$

$$h(t) = \begin{cases} \dfrac{1}{t} - \dfrac{1}{e^t - 1}, & t \neq 0; \\ \dfrac{1}{2}, & t = 0; \end{cases} \tag{2}$$

and, for $a \in \mathbb{R}$,

$$F_a(t) = \begin{cases} \dfrac{t}{e^{at} - e^{(a-1)t}}, & t \neq 0; \\ 1, & t = 0. \end{cases} \tag{3}$$

In [3, p. 217] and [7, p. 295 and p. 704], finding the best bounds for the function $f(x)$ on $(0, 1)$ was ever proposed as an open problem. In recent years, this open problem was investigated by several mathematicians in [2, 4, 8, 9, 19] and related references

*Mathematics Subject Classifications: 26A48, 26A51, 33B15.

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therein. Recently, the open problem was concluded by [8, Theorem 1]: The function $f(x)$ defined by (1) is strictly decreasing on $(0,\infty)$, with

$$\lim_{x\to 0^+} f(x) = \frac{1}{12} \quad \text{and} \quad \lim_{x\to\infty} f(x) = 0. \tag{4}$$

It is known that the function $h(t)$ is related with Binet's first formula for the gamma function $\Gamma(x)$ by

$$\ln\Gamma(x) = \left(x - \frac{1}{2}\right)\ln x - x + \ln\sqrt{2\pi} + \int_0^\infty \left[\frac{1}{2} - h(t)\right]\frac{e^{-xt}}{t}\,\mathrm{d}t, \quad x > 0. \tag{5}$$

The integral in (5) is called the remainder of Binet's first formula for the logarithm of the gamma function. In [6], some properties of the function $h(t)$ on $(0,\infty)$ were presented and applied to study the completely monotonic properties of the difference between remainders of Binet's first formula (5).

In [8], the very possible origin and background of the function $f(t)$ and the open problem above were reasoned and the relationships among $f(t)$, $h(t)$ and $F_a(t)$,

$$h'(t) = -f(t), \quad [\ln F_a(t)]' = h(t) - a \quad \text{and} \quad [\ln F_a(t)]'' = -f(t), \tag{6}$$

were partially or implicitly remarked.

In fact, the relationships in (6) were unaware before the paper [8], to the best of our knowledge. These relationships connect three seemingly unrelated problems or functions, especially the remainder of Binet's first formula for the logarithm of the gamma function. Therefore, it may be significant to systematically explore these three functions.

In [8, Theorem 1 and Theorem 3], the following results were procured by spending almost two pages through complex and unattractive arguments: The function $f(t)$ defined by (1) is strictly decreasing on $(0,\infty)$; the function $h(t)$ defined by (2) is decreasing on $(0,\infty)$; the function $F_a(t)$ defined by (3) is logarithmically concave on $(0,\infty)$; when $a \ge \frac{1}{2}$, the function $F_a(t)$ is decreasing on $(0,\infty)$.

DEFINITION 1. A positive and k-times differentiable function $f(x)$ is said to be k-log-convex (or k-log-concave, respectively) on an interval I with $r \ge 2$ if and only if $[\ln f(x)]^{(k)} \ge 0$ (or $[\ln f(x)]^{(k)} \le 0$, respectively) on I.

In [4, Theorem 1 and Theorem 3], by the celebrated Hermite-Hadamard's integral inequality (see [15, 18]) for convex functions and the power series expansion of e^t at $t = 0$, an awkward proof for equivalent forms of the following Theorem 1 which extended [8, Theorem 1 and Theorem 3] were provided by spending almost two pages.

The aim of this note is to supply a concise proof for monotonic and logarithmically convex properties of functions $f(t)$, $h(t)$ and $F_a(t)$.

THEOREM 1. The function $F_a(t)$ is decreasing on $\mathbb{R}$ if $a \ge 1$, increasing on $\mathbb{R}$ if $a \le 0$, increasing on $(-\infty,0)$ if $a \le \frac{1}{2}$, and decreasing on $(0,\infty)$ if $a \ge \frac{1}{2}$.

The function $F_a(t)$ for $a \in \mathbb{R}$ is logarithmic concave on $\mathbb{R}$. Equivalently, the function $h(t)$ is decreasing and $f(t)$ is positive and even on $\mathbb{R}$.

The function $F_a(t)$ for $a \in \mathbb{R}$ is 3-log-concave on $(-\infty,0)$ and 3-log-convex on $(0,\infty)$. Equivalently, the function $h(t)$ is concave on $(-\infty,0)$ and convex on $(0,\infty)$, and the function $f(t)$ is increasing on $(-\infty,0)$ and decreasing on $(0,\infty)$.

After proving Theorem 1 concisely in next section, we will give in the final section some remarks for explaining the novelty of the proof for Theorem 1.

2 A Concise Proof of Theorem 1

For $t \neq 0$, taking the logarithm of $F_a(t)$ and differentiating yield

$$\begin{aligned}\ln F_a(t) &= \ln|t| - \ln\left|1-e^{-t}\right| - at,\\ [\ln F_a(t)]' &= \frac{1}{t} - \frac{1}{e^t-1} - a = h(t) - a,\\ [\ln F_a(t)]'' = h'(t) &= \frac{e^t}{(e^t-1)^2} - \frac{1}{t^2} = -f(t)\end{aligned}$$

and

$$\begin{aligned}[\ln F_a(t)]^{(3)} &= \frac{2}{t^3} - \frac{e^t(1+e^t)}{(e^t-1)^3}\\ &= \frac{2e^{-3t/2}}{t^3}\left(\frac{-t}{e^{-t}-1}\right)^3\left[\left(\frac{e^{-t/2}-e^{t/2}}{-t}\right)^3 - \frac{e^{-t/2}+e^{t/2}}{2}\right]\\ &= \frac{2e^{-3t/2}}{t^3}\left(\frac{-t}{e^{-t}-1}\right)^3\left\{\left[\frac{\sinh(-t/2)}{-t/2}\right]^3 - \cosh\left(-\frac{t}{2}\right)\right\}.\end{aligned}$$

Lazarević's inequality collected in [1, p. 131] and [7, p. 300] states that

$$\left(\frac{\sinh t}{t}\right)^3 > \cosh t \tag{7}$$

for $t \neq 0$. Hence, it directly follows that $[\ln F_a(t)]^{(3)}$ is negative on $(-\infty,0)$ and positive on $(0,\infty)$, that is, the function $F_a(t)$ for $a \in \mathbb{R}$ is 3-log-concave on $(-\infty,0)$ and 3-log-convex on $(0,\infty)$. This means that the function $[\ln F_a(t)]'' = -f(t)$ is decreasing on $(-\infty,0)$ and increasing on $(0,\infty)$. Since $\lim_{t\to\pm\infty} f(t) = 0$ is easy to see, then $[\ln F_a(t)]'' < 0$. Consequently, the function $[\ln F_a(t)]' = h(t) - a$ is decreasing on $\mathbb{R}$. Since it is immediate that $\lim_{t\to-\infty} h(t) = 1$ and $\lim_{t\to\infty} h(t) = 0$, the function $[\ln F_a(t)]'$ is negative on $\mathbb{R}$ if $a \geq 1$, positive on $\mathbb{R}$ if $a \leq 0$, positive on $(-\infty,0)$ if $a \leq \frac{1}{2}$, and negative on $(0,\infty)$ if $a \geq \frac{1}{2}$. The proof of Theorem 1 is complete.

3 Remarks

Now we would like to demonstrate why the proof of Theorem 1 is novel.

REMARK 1. The first key step in the proof of Theorem 1 is to make use of Lazarević's inequality (7) listed in [1, p. 131] and [7, p. 300]. The second key step is to take the logarithm of the function $F_a(t)$.

REMARK 2. Observing that there is a parameter a in the definition of $F_a(t)$. If replacing $a-1$ by b, then the function $F_a(t)$ can be generalized as

$$F_{a,b}(t)=\begin{cases}\dfrac{t}{e^{bt}-e^{at}}, & t\ne 0\\ \dfrac{1}{b-a}, & t=0\end{cases} \tag{8}$$

for real numbers a and b with $b>a$.

REMARK 3. Along the route of the proof of Theorem 1, the following properties of $F_{a,b}(t)$ were shown in [14].

THEOREM 2 ([14, Theorem 1]). For real numbers a and b with $b>a$, the function $F_{a,b}(t)$ is 3-log-concave on $(-\infty,0)$, 3-log-convex on $(0,\infty)$, logarithmic concave on $(-\infty,\infty)$, and the function

$$H_{a,b}(t)=\begin{cases}\dfrac{1}{t}-\dfrac{be^{bt}-ae^{at}}{e^{bt}-e^{at}}, & t\ne 0\\ -\dfrac{a+b}{2}, & t=0\end{cases} \tag{9}$$

is decreasing on $(-\infty,\infty)$ with

$$\lim_{t\to-\infty}H_{a,b}(t)=-a \quad\text{and}\quad \lim_{t\to\infty}H_{a,b}(t)=-b. \tag{10}$$

REMARK 4. If replacing $h(t)$ in (5) by $H_{a,b}(t)$ defined in (9), then the remainder of Binet's first formula for the logarithm of the gamma function may be naturally extended. For detailed information, please refer to [13].

REMARK 5. For positive numbers x and y with $y>x$, set

$$g_{x,y}(t)=\int_x^y u^{t-1}\,\mathrm{d}u=\begin{cases}\dfrac{y^t-x^t}{t}, & t\ne 0;\\ \ln y-\ln x, & t=0.\end{cases} \tag{11}$$

It is clear that

$$F_{a,b}(t)=\frac{1}{g_{e^b,e^a}(t)} \quad\text{and}\quad g_{x,y}(t)=\frac{1}{F_{\ln x,\ln y}(t)}. \tag{12}$$

Some properties of $g_{x,y}(t)$ have been investigated in [16, 17]. From Theorem 2 and identities in (12), some new properties of $g_{x,y}(t)$ can be derived as follows.

THEOREM 3 ([14, Theorem 2]). For positive numbers x and y with $y>x$, the function $g_{x,y}(t)$ is 3-log-convex on $(-\infty,0)$, 3-log-concave on $(0,\infty)$, logarithmic convex on $(-\infty,\infty)$, and the function

$$h_{x,y}(t)=\begin{cases}\dfrac{y^t\ln y-x^t\ln x}{y^t-x^t}-\dfrac{1}{t}, & t\ne 0\\ \ln\sqrt{xy}\,, & t=0\end{cases} \tag{13}$$

is increasing on $(-\infty, \infty)$ with

$$\lim_{t\to-\infty} h_{x,y}(t) = \ln x \quad \text{and} \quad \lim_{t\to\infty} h_{x,y}(t) = \ln y. \tag{14}$$

REMARK 6. The properties of $h_{x,y}(t)$ in Theorem 3 have been applied in [5, 10, 12] to simplify the proofs of the monotonic, logarithmically convex, and Schur-convex properties of Stolarsky's mean values $E(r,s;x,y)$ defined by

$$\begin{aligned}
&E(r,s;x,y) = \left(\frac{r}{s}\cdot\frac{y^s - x^s}{y^r - x^r}\right)^{1/(s-r)}, && rs(r-s)(x-y) \neq 0;\\
&E(r,0;x,y) = \left(\frac{1}{r}\cdot\frac{y^r - x^r}{\ln y - \ln x}\right)^{1/r}, && r(x-y) \neq 0;\\
&E(r,r;x,y) = \frac{1}{e^{1/r}}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)}, && r(x-y) \neq 0;\\
&E(0,0;x,y) = \sqrt{xy}, && x \neq y;\\
&E(r,s;x,x) = x, && x = y;
\end{aligned} \tag{15}$$

where x and y are positive numbers and $r, s \in \mathbb{R}$.

REMARK 7. The properties of $h_{x,y}(t)$ in Theorem 3 have also been applied in [12] to yield some new monotonic and logarithmically convex properties of Stolarsky's mean values $E(r,s;x,y)$. These new properties of $E(r,s;x,y)$ are similar to those obtained in [11] for Gini's mean values $G(r,s;x,y)$ defined by

$$G(r,s;x,y) = \begin{cases} \left(\dfrac{x^s + y^s}{x^r + y^r}\right)^{1/(s-r)}, & r \neq s;\\[2ex] \exp\left(\dfrac{x^r \ln x + y^r \ln y}{x^r + y^r}\right), & r = s \neq 0; \end{cases} \tag{16}$$

where x and y are positive variables and r and s are real variables.

REMARK 8. In conclusion, the proof of Theorem 1 is not only concise but also novel.

Acknowledgment. The third author was partially supported by the China Scholarship Council.

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