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Generating Some Semiclassical Orthogonal Polynomials*

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Abstract

By means of the quadratic decomposition, this research confirms that if v is a regular symmetric semiclassical form (linear functional), then the form u defined by the relation $(x-a)xu = -\lambda v$ is also regular and semiclassical form for every complex λ except for a discrete set of numbers depending on a^2 and the even part of v. An example related to the Generalized Gegenbauer form is worked out.

1 Introduction

Let v be a regular form. We define a new form u by the relation $D(x)u = A(x)v$ where $A(x)$ and $D(x)$ are non-zero polynomials. An extensive study of the form u has been carried out by several authors from different sides [2, 5, 6, 7, 9, 12, 14]. In particular, in [9] and for $A(x) = -\lambda \neq 0$ and $D(x) = x^2$, Maroni found necessary and sufficient conditions for u to be regular. Also, an explicit expression for the orthogonal polynomials (O.P.) with respect to u is given. Finally, it was proved that if v is a semiclassical form (see [1, 3]), then u is also a semiclassical form. The following aims to study the form u, fulfilling

$$(x-a)xu = -\lambda v, \quad \lambda \neq 0, \ \ (u)_1 = a \neq 0,$$

where v is a regular symmetric form. We prove that the regularity of the new form u depends only on a^2 and the even part of v . The coefficients of the three-term recurrence relation satisfied by the corresponding sequence of O.P., are given explicitly. The stability of the semiclassical families is proved. At last, we apply our results to Generalized Gegenbauer form.

2 Preliminaries and Notations

Let $\mathcal{P}$ be the vector space of polynomials with coefficients in $\mathbb{C}$ and let $\mathcal{P}'$ be its dual. We denote by $\langle u, f\rangle$ the action of $u \in \mathcal{P}'$ on $f \in \mathcal{P}$. In particular, we denote by $(u)_n := \langle u, x^n\rangle, n \geq 0$, the moments of u. For any form u and any polynomial h let $Du = u'$, hu, δ_0, and $(x-c)^{-1}u$ be the forms defined by: $\langle u', f\rangle :=$

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$-\langle u, f'\rangle$, $\langle hu, f\rangle := \langle u, hf\rangle$, $\langle \delta_c, f\rangle := f(c)$, and $\left\langle (x-c)^{-1}u, f\right\rangle := \langle u, \theta_c f\rangle$ where $(\theta_c f)(x) = \frac{f(x)-f(c)}{x-c}$, $c \in \mathbb{C}$, $f \in \mathcal{P}$.

Then, it is straightforward to prove that for $f \in \mathcal{P}$ and $u \in \mathcal{P}'$, we have

$$(x-c)^{-1}\left((x-c)u\right) = u - (u)_0\delta_c \ , \tag{1}$$

$$(fu)' = f'u + fu' \ . \tag{2}$$

The form v will be called regular if we can associate with it a sequence $\{S_n\}_{n\geq 0}$ $\big(\deg(S_n) \leq n\ \big)$ such that

$$\langle v, S_n S_m\rangle = r_n \delta_{n,m} \ , \quad n, m \geq 0, \quad r_n \neq 0, \quad n \geq 0.$$

Then $\deg(S_n) = n$, $n \geq 0$, and we can always suppose each S_n monic (i.e. $S_n(x) = x^n + \cdots$). The sequence $\{S_n\}_{n\geq 0}$ is said to be orthogonal with respect to v. It is a very well known fact that the sequence $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation $\big($see, for instance, the monograph by Chihara [4]$\big)$

$$\begin{array}{l} S_{n+2}(x) = (x-\xi_{n+1})S_{n+1}(x) - \rho_{n+1}S_n(x) \ , \quad n \geq 0 \ , \\ S_1(x) = x - \xi_0 \ , \qquad S_0(x) = 1 \ , \end{array} \tag{3}$$

with $(\xi_n, \rho_{n+1}) \in \mathbb{C} \times \mathbb{C} - \{0\}$, $n \geq 0$, by convention we set $\rho_0 = (v)_0 = 1$.

In this case, let $\{S_n^{(1)}\}_{n\geq 0}$ be the associated sequence of first kind for the sequence $\{S_n\}_{n\geq 0}$ satisfying the three-term recurrence relation

$$\begin{array}{l} S_{n+2}^{(1)}(x) = (x-\xi_{n+2})S_{n+1}^{(1)}(x) - \rho_{n+2}S_n^{(1)}(x) \ , \quad n \geq 0, \\ S_1^{(1)}(x) = x - \xi_1, \qquad S_0^{(1)}(x) = 1 \ , \quad \Big(S_{-1}^{(1)}(x) = 0\Big) \ . \end{array} \tag{4}$$

Another important representation of $S_n^{(1)}(x)$ is, (see [4]),

$$S_n^{(1)}(x) := \left\langle v, \frac{S_{n+1}(x) - S_{n+1}(\zeta)}{x-\zeta} \right\rangle \ , \quad n \geq 0. \tag{5}$$

Also, let $\{S_n(., \mu)\}_{n\geq 0}$ be the co-recursive polynomials for the sequence $\{S_n\}_{n\geq 0}$ satisfying [4]

$$S_n(x, \mu) = S_n(x) - \mu S_{n-1}^{(1)}(x), \quad n \geq 0. \tag{6}$$

A form v is called symmetric if $(v)_{2n+1} = 0, n \geq 0$. The conditions $(v)_{2n+1} = 0, n \geq 0$ are equivalent to the fact that the corresponding monic orthogonal polynomials sequence (MOPS) $\{S_n\}_{n\geq 0}$ satisfies the recurrence relation (3) with $\xi_n = 0, n \geq 0$ [4].

3 Algebraic Properties

In this paper, unless stated otherwise, the form v will be supposed to be normalized, $\big($i.e: $(v)_0 = 1\big)$, symmetric, and regular and $\{S_n\}_{n\geq 0}$ be the corresponding MOPS. For fixed $a \in \mathbb{C}$ and $\lambda \in \mathbb{C} - \{0\}$, we can define a new normalized form $u \in \mathcal{P}'$ by the relation

$$(x-a)xu = -\lambda v \ \ , \ \ (u)_1 = a \ . \tag{7}$$

Equivalently, from (1) we have

$$u = -\lambda(x-a)^{-1}x^{-1}v + \delta_a = -\lambda x^{-1}(x-a)^{-1}v + \delta_a. \tag{8}$$

The case $a = 0$ is treated in [1, 9, 13], so henceforth, we assume $a \neq 0$.

When u is regular, let $\{Z_n\}_{n\geq 0}$ be its corresponding MOPS. It satisfies

$$\begin{aligned} &Z_{n+2}(x) = (x-\beta_{n+1})\, Z_{n+1}(x) - \gamma_{n+1} Z_n(x), \quad n \geq 0, \\ &Z_1(x) = x - \beta_0, \qquad Z_0(x) = 1. \end{aligned} \tag{9}$$

From (7), the sequence $\{Z_n\}_{n\geq 0}$, when it exists, satisfies the following finite-type relation [8, p.301, Proposition 2.1.]

$$\begin{aligned} &Z_{n+2}(x) = S_{n+2}(x) + b_{n+1}S_{n+1}(x) + a_n S_n(x), \quad n \geq 0, \\ &Z_1(x) = S_1(x) + b_0 S_0(x), \end{aligned} \tag{10}$$

with $(a_n, b_n) \in (\mathbb{C} - \{0\}) \times \mathbb{C}$. In this condition, the sequence $\{Z_n\}_{n\geq 0}$ is orthogonal with respect to u if and only if

$$\langle u, Z_{n+1}\rangle = 0, \quad \langle u, xZ_{n+2}\rangle = 0, \quad n \geq 0, \quad \langle u, xZ_1\rangle \neq 0.$$

Consequently $\langle u, xZ_{n+2}\rangle = 0$ and $\langle u, (x-a)Z_{n+2}\rangle = 0$, $n \geq 0$. So, using (5), (8) and (10), we obtain respectively

$$\Big(aS_{n+1}(a) - \lambda S_n^{(1)}(a)\Big)\, b_{n+1} + \Big(aS_n(a) - \lambda S_{n-1}^{(1)}(a)\Big)\, a_n = \lambda S_{n+1}^{(1)}(a) - aS_{n+2}(a) \tag{11}$$

and

$$S_n^{(1)}(0) b_{n+1} + S_{n-1}^{(1)}(0) a_n = -S_{n+1}^{(1)}(0). \tag{12}$$

The determinant of the system defined by (11)-(12) is

$$\triangle_n = S_{n-1}^{(1)}(0)\left(aS_{n+1}(a) - \lambda S_n^{(1)}(a)\right) - S_n^{(1)}(0)\left(aS_n(a) - \lambda S_{n-1}^{(1)}(a)\right),\ n \geq 0\,. \tag{13}$$

Let us recall some general features. Consider the quadratic decomposition of $\{S_n\}_{n\geq 0}$ and $\{S_n^{(1)}\}_{n\geq 0}$ [10]

$$S_{2n}(x) = P_n(x^2), \quad S_{2n+1}(x) = xR_n(x^2), \quad n \geq 0, \tag{14}$$

$$S_{2n}^{(1)}(x) = R_n(x^2, -\rho_1), \quad S_{2n+1}^{(1)}(x) = xP_n^{(1)}(x^2), \quad n \geq 0. \tag{15}$$

The sequences $\{P_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ are respectively orthogonal with respect to σv and $x\sigma v$ where σv is the even part of v defined by $\langle \sigma v, f\rangle := \langle v, (\sigma f)(x)\rangle = \langle v, f(x^2)\rangle$. We have for instance:

$$\begin{aligned} &P_{n+2}(x) = (x - \rho_{2n+2} - \rho_{2n+3})P_{n+1}(x) - \rho_{2n+1}\rho_{2n+2}P_n(x), \quad n \geq 0, \\ &P_1(x) = x - \rho_1, \quad P_0(x) = 1 \end{aligned} \tag{16}$$

Using (14)-(15), (13) becomes

$$\triangle_{2n} = -aS_{2n}^{(1)}(0)P_n(a^2, \lambda), \quad \triangle_{2n+1} = -aS_{2n}^{(1)}(0)P_{n+1}(a^2, \lambda), \quad n \geq 0. \tag{17}$$

Taking into account (4), with $\xi_n = 0$, we get $S^{(1)}_{n+2}(0) = -\rho_{n+2}S^{(1)}_n(0)$. Then,

$$S^{(1)}_{2n+2}(0) = (-1)^{n+1}\prod_{\nu=0}^{n}\rho_{2\nu+2} \neq 0, \quad n \geq 0.$$

Consequently, we easily deduce the following result:

PROPOSITION 1. The form u is regular if and only if $P_n(a^2, \lambda) \neq 0,\ n \geq 0$, where P_n is defined by (14).

REMARKS (1.) When v is symmetric and positive definite, $\lambda \in \mathbb{R} - \{0\}$, then u is regular for every a such that $a^2 \notin \mathbb{R}$. (2.) u is regular if and only if $\lambda \neq \lambda_n, n \geq 0$ where $\lambda_n = \frac{P_n(a^2)}{P^{(1)}_{n-1}(a^2)}$.

When $\triangle_n \neq 0, n \geq 0$, by solving the system (11)-(12), we obtain

$$a_{2n} = -\frac{P_{n+1}(a^2, \lambda)}{P_n(a^2, \lambda)},\ a_{2n+1} = \rho_{2n+2},\ n \geq 0, \tag{18}$$

$$b_{2n} = -a, \quad b_{2n+1} = 0, \quad n \geq 0. \tag{19}$$

PROPOSITION 2. We may write

$$\gamma_{n+2} = \frac{a_n}{a_{n-1}}\rho_n, \quad n \geq 1, \tag{20}$$

$$\beta_{n+1} = b_n - b_{n+1}, \quad n \geq 0, \tag{21}$$

$$\gamma_{n+2} = \rho_{n+2} - b_{n+1}\beta_{n+2} + a_n - a_{n+1}, \quad n \geq 0, \tag{22}$$

$$b_n\gamma_{n+2} = b_{n+1}\rho_{n+1} - a_n\beta_{n+2}, \quad n \geq 0. \tag{23}$$

PROOF. After multiplication of (10) by x, we apply the recurrence relations (3) and (9) we get

$$\begin{aligned} Z_{n+3} + \beta_{n+2}Z_{n+2} + \gamma_{n+2}Z_{n+1} \quad = \quad & S_{n+3} + \rho_{n+2}S_{n+1} \\ & + b_{n+1}S_{n+2} + b_{n+1}\rho_{n+1}S_n + a_nS_{n+1} + a_n\rho_nS_{n-1}. \end{aligned}$$

Substituting Z_{k+2} in the above equation by $S_{k+2}+b_{k+1}S_{k+1}+a_kS_k$ with $k = n+1, n, n-1$, we obtain (20)-(23), after comparing the coefficients of S_k with $n-1 \leq k \leq n+2$.

COROLLARY 1. We have

$$\begin{cases} \beta_n = (-1)^n a, \\ \gamma_1 = -\lambda\ , \quad \gamma_{2n+2} = a_{2n}\ , \quad \gamma_{2n+3} = \frac{\rho_{2n+1}\rho_{2n+2}}{a_{2n}}\ , \quad n \geq 0\ . \end{cases} \tag{24}$$

PROOF. Using (20)-(22) and taking (18)-(19) into account, it's quite straightforward to get the expressions of β_n and γ_{n+1} for $n \geq 1$. From $\langle u, Z_k \rangle = 0,\ \ 1 \leq k \leq 2$, we obtain $\beta_0 = a$ and $\gamma_1 = -\lambda$, by virtue of (7). Hence (24).

REMARK. Since $a \neq 0$, on account of (24) the form u is not symmetric.

4 The Semiclassical Case

Let us recall that a form v is called semiclassical when it is regular and there exist two polynomials Φ and Ψ such that:

$$(\Phi v)' + \Psi v = 0 \ , \quad \deg(\Psi) \geq 1 \ , \quad \Phi \ \text{ monic.} \tag{25}$$

The class of the semiclassical form v is $s = \max(\deg \Psi - 1, \deg \Phi - 2)$ if and only if the following condition is satisfied

$$\prod_c \left(|\Phi'(c) + \Psi(c)| + \left|\left\langle u, \theta_c \Psi + \theta_c^2 \Phi \right\rangle\right| \right) > 0 \ , \tag{26}$$

where c goes over the roots set of Φ [11].

In the sequel the form v will be supposed symmetric and semiclassical of class s satisfying (25). From (7), and (25), it is clear that the form u, when it is regular, it is also semiclassical and satisfies

$$\left(\tilde{\Phi} u\right)' + \tilde{\Psi} u = 0 \tag{27}$$

with

$$\tilde{\Phi}(x) = x(x-a)\Phi(x) \ , \quad \tilde{\Psi}(x) = x(x-a)\Psi(x) \ . \tag{28}$$

The class $\tilde{s}$ of u is at most $s+2$.

PROPOSITION 3. The class of u depends only on the zeros $x = 0$ and $x = a$.

PROOF. Let c be a root of $\tilde{\Phi}$ such that $c \in \mathbb{C} - \{0, a\}$, then $\Phi(c) = 0$. If $\Phi'(c) + \Psi(c) \neq 0$, using (28) we have $\tilde{\Phi}'(c) + \tilde{\Psi}(c) = c(c-a)\left(\Phi'(c) + \Psi(c)\right) \neq 0$. If $\Phi'(c) + \Psi(c) = 0$, we have $c(c-a)\left(\Phi'(c) + \Psi(c)\right) = x(x-a)\left(\Phi'(c) + \Psi(c)\right) = 0$, which leads to $\theta_c^2 \tilde{\Phi} + \theta_c \tilde{\Psi} = x(x-a)\left(\theta_c^2 \Phi + \theta_c \Psi\right)$. Then, using (7) and the above result, we get $\left\langle u, \theta_c^2 \tilde{\Phi} + \theta_c \tilde{\Psi} \right\rangle = -\lambda \left\langle v, \theta_c^2 \Phi + \theta_c \Psi \right\rangle \neq 0$ according to (26). In any case, we cannot simplify by $x - c$.

THEOREM 1. Let v be a semiclassical form of class s, satisfying (25), $c \in \{0, a\}$, $X_1(c) = | \ \Phi(c) \ | + \left| -\lambda \left\langle v, \theta_c \Psi + \theta_c^2 \Phi \right\rangle + \Phi(c) + c(\Phi'(c) + \Psi(c)) \right|$, and

$$X_2(c) = | \ 2\Phi'(c) + \Psi(c) \ | + \left| -\lambda \left\langle v, \theta_c^2 \Psi + \theta_c^3 \Phi \right\rangle + 2\Phi'(c) + \Psi(c) + c\left(2\Phi''(c) + \Psi'(c)\right) \right| .$$

For every $a, \lambda \in \mathbb{C} - \{0\}$ such that $P_n(a^2, \lambda) \neq 0, n \geq 0$, the form u defined by (7) is regular and semiclassical of class $\tilde{s}$ satisfying (27)-(28). Moreover,

1) If $X_1(a)X_1(0) \neq 0$ then $\tilde{s} = s + 2$,

2) If $X_1(0) \neq 0$, $X_1(a) = 0$ and $X_2(a) \neq 0$ or $X_1(a) \neq 0$, $X_1(0) = 0$ and $X_2(0) \neq 0$ then $\tilde{s} = s + 1$,

3) If $X_1(a)X_1(0) = 0$ and $X_2(a)X_2(0) \neq 0$ then $\tilde{s} = s$.

PROOF. From (28), we have $\tilde{\Phi}'(a) + \tilde{\Psi}(a) = a\Phi(a)$ and $\tilde{\Phi}'(0) + \tilde{\Psi}(0) = -a\Phi(0)$. Using (8), we get $\left\langle u, \theta_a \tilde{\Psi} + \theta_a^2 \tilde{\Phi} \right\rangle = -\lambda \left\langle v, \theta_a \Psi + \theta_a^2 \Phi \right\rangle + \Phi(a) + a(\tilde{\Phi}'(a) + \Psi(a))$ and $\left\langle u, \theta_0 \tilde{\Psi} + \theta_0^2 \tilde{\Phi} \right\rangle = -\lambda \left\langle v, \theta_0 \Psi + \theta_0^2 \Phi \right\rangle + \Phi(0)$. Hence $X_1(c), c \in \{a, 0\}$.

1) If $X_1(a)X_1(0) \neq 0$, then it is not possible to simplify according to the standard criterion (26), which means that the class of u is $\tilde{s} = s + 2$.

2) If $X_1(0) \neq 0$ and $X_1(a) = 0$, then it is only possible to simplify by $x - a$. Then, u fulfils (27) with

$$\tilde{\Phi}(x) = x\Phi(x)\ , \quad \tilde{\Psi}(x) = x\left(\theta_a\Phi(x) + \Psi(x)\right)\ . \tag{29}$$

Here, we have $\tilde{\Phi}'(a) + \tilde{\Psi}(a) = a\left(2\Phi'(a) + \Psi(a)\right)$. Using (8) and the definition of the operator θ_a, we obtain

$$\left\langle u, \theta_a\tilde{\Psi} + \theta_a^2\tilde{\Phi}\right\rangle = -\lambda\left\langle v, \theta_a^2\Psi + \theta_a^3\Phi\right\rangle + 2\Phi'(a) + \Psi(a) + a\left(2\Phi'(a) + \Psi'(a)\right).$$

Since $a \neq 0$, if $X_2(a)) \neq 0$, then it is not possible to simplify, which means that the class of u is $\tilde{s} = s + 1$. Using the same proceeding, we easily prove that, if $X_1(a) \neq 0$, $X_1(0) = 0$ and $X_2(0)) \neq 0$, then u fulfils (27) with

$$\tilde{\Phi}(x) = (x - a)\Phi(x)\ , \quad \tilde{\Psi}(x) = (x - a)\left(\theta_0\Phi(x) + \Psi(x)\right)\ . \tag{30}$$

and, $\tilde{s} = s + 1$.

3) If $X_1(a)X_1(0) = 0$, we can simplify (27)-(28) by $x(x - a)$.We obtain

$$\tilde{\Phi}(x) = \Phi(x)\ , \quad \tilde{\Psi}(x) = \theta_0\Phi(x) + \theta_a\Phi(x) + \Psi(x)\ . \tag{31}$$

Then, $\tilde{s} = s$ if $X_2(a)X_2(0) \neq 0$.

Finally, if we suppose that the form v has the following integral representation:

$$\langle v, f\rangle = \int_{-\infty}^{+\infty} V(x)f(x)dx\ , \quad f \in \mathcal{P}, \quad \text{with} \quad (v)_0 = \int_{-\infty}^{+\infty} V(x)dx = 1$$

where V is a locally integrable function with rapid decay and continuous at the origin and the point $x = a$. Then, from (8) and after some straightforward computations, we prove that, the form u is represented by

$$\begin{aligned}\langle u, f\rangle \quad &= \quad f(a)\left\{1 + \frac{\lambda}{a}P\int_{-\infty}^{+\infty}\frac{V(x)}{x - a}dx\right\} - \\ &\quad -\frac{\lambda}{a}P\int_{-\infty}^{+\infty}\frac{V(x)}{x - a}f(x)dx + \frac{\lambda}{a}P\int_{-\infty}^{+\infty}\frac{V(x)}{x}f(x)dx\ ,\end{aligned} \tag{32}$$

where for $c \in \{0, a\}$

$$P\int_{-\infty}^{+\infty}\frac{V(x)}{x - c}f(x)dx = \lim_{\epsilon \longrightarrow 0}\left\{\int_{-\infty}^{c-\epsilon}\frac{V(x)}{x - c}f(x)dx + \int_{c+\epsilon}^{+\infty}\frac{V(x)}{x - c}f(x)dx\right\}.$$

5 Application

Theorem 1 shows that we can generate new semiclassical sequences from well known ones. We apply our results to $v := GG$, where GG is the Generalized Gegenbauer form. In this case, the form v is symmetric semiclassical of class $s = 1$. Thus, we have [4]

$$\begin{cases} \rho_{2n+1} = \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \\ \rho_{2n+2} = \frac{(n+1)(n+\alpha+1)}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)},\ n \geq 0. \end{cases} \tag{33}$$

The regularity conditions are $\alpha \neq -n$, $\beta \neq -n$, $\alpha+\beta \neq -n$, $n \geq 1$. We also have

$$\Phi(x) = x(x^2-1), \quad \Psi(x) = -2(\alpha+\beta+2)x^2 + 2(\beta+1). \tag{34}$$

For greater convenience we take $a = 1$, and $\alpha \neq 0$. From (16) and (33), we can easily obtain by induction

$$P_n(1) = S_{2n}(1) = \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(2n+\alpha+\beta+1)} \ , n \geq 0 \ . \tag{35}$$

From (16) and (4), we get the recurrence relation satisfied by $\{P_n^{(1)}\}_{n\geq 0}$. Using this relation and (33), we deduce by induction

$$\begin{aligned} P_n^{(1)}(1) = \tfrac{-(\alpha+\beta+1)}{\alpha\Gamma(2n+\alpha+\beta+3)} \Bigg(& \tfrac{\Gamma(\alpha+\beta+1)\Gamma(n+2)\Gamma(n+\beta+2)}{\Gamma(\beta+1)} - \\ & - \tfrac{\Gamma(n+\alpha+2)\Gamma(n+\alpha+\beta+2)}{\Gamma(\alpha+1)} \Bigg) \ , \ n \geq 0 \ . \end{aligned} \tag{36}$$

From (6) and (35)-(36), we get

$$P_n(1,\lambda) = \frac{(\alpha+\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)}{\alpha\Gamma(\alpha+1)\Gamma(2n+\alpha+\beta+1)} d_n \ , n \geq 0 \ , \tag{37}$$

with

$$d_n = \lambda \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta+1)\Gamma(n+1)\Gamma(n+\beta+1)}{\Gamma(\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)} + \frac{\alpha}{\alpha+\beta+1} - \lambda \ . \tag{38}$$

Then, u is regular for every $\lambda \neq 0$ such that

$$\lambda^{-1} - \frac{\alpha+\beta+1}{\alpha} \neq \frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta+2)\Gamma(n+1)\Gamma(n+\beta+1)}{\alpha\Gamma(\beta+1)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+1)} \ , n \geq 0 \ .$$

Now, we give the coefficients of the recurrence relation satisfied by $\{Z_n\}_{n\geq 0}$. For this, first we calculate the coefficients a_n and b_n, $n \geq 0$ given by (18)-(19).

$$\begin{cases} a_{2n} = -\frac{(n+1)(n+\alpha+1)d_{n+1}}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)d_n}, \ a_{2n+1} = \rho_{2n+2} \ , \\ b_{2n} = -1, \quad b_{2n+1} = 0, \quad n \geq 0. \end{cases} \tag{39}$$

Using the above results and (24), we obtain

$$\begin{cases} \gamma_{2n+2} = a_{2n} \quad , \quad \gamma_{2n+3} = -\frac{(n+1)(n+\beta+1)d_n}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+3)d_{n+1}}, \\ \beta_n = (-1)^n, \quad \gamma_1 = -\lambda, \quad n \geq 0. \end{cases} \tag{40}$$

Since v is semiclassical, then according to Theorem 1. the form u is also semiclassical. It satisfies (27) with

$$\tilde{\Phi}(x) = x(x-1)^2(x+1) \ , \quad \tilde{\Psi}(x) = x(x-1)\left(-2(\alpha+\beta+2)x^2 + 2(\beta+1)\right). \tag{41}$$

After some calculations and taking account of (34), we obtain $X_1(0) = 0$, $X_1(1) = 2 \mid \lambda(\alpha+\beta+1) - \alpha \mid$, $X_2(0) = 2 \mid \beta \mid +2 \mid -\lambda(\alpha+\beta+1) + \beta \mid$ and $X_2(1) = 2 \mid \alpha \mid +4 \mid \alpha+\beta-1 \mid$ (we take $\lambda = \frac{\alpha}{\alpha+\beta+1}$ in calculation of $X_2(1)$). Now, it is enough to use Theorem 1. in order to obtain the following results:

$\star$ If $\lambda \neq \frac{\alpha}{\alpha+\beta+1}$, then the class of u is $\tilde{s} = 2$.

$\star$ If $\lambda = \frac{\alpha}{\alpha+\beta+1}$, then the class of u is $\tilde{s} = 1$.

The form v has the following integral representation [4 p.156], for $\Re\alpha > -1$, $\Re\beta > -1$, $f \in \mathcal{P}$,

$$\langle v, f\rangle = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} |x|^{2\beta+1}(1-x^2)^{\alpha} f(x)dx. \tag{42}$$

Therefore, for $\Re\beta > -\frac{1}{2}$, $\Re\alpha > 0$, $f \in \mathcal{P}$, (32) becomes

$$\begin{aligned}\langle u, f\rangle &= \left(1 - \frac{\lambda(\alpha+\beta+1)}{\alpha}\right) f(1) \\ &\quad + \frac{\lambda\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)} \int_{-1}^{1} sgn(x)|x|^{2\beta}(1+x)^{\alpha}(1-x)^{\alpha-1} f(x)dx.\end{aligned}$$

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