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Approximate Solutions Of The Forced Duffing Equation With Mixed Nonlinearities*

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Abstract

In this paper, an improved form of the generalized quasilinearization technique is developed to obtain a monotone sequence of approximate solutions converging uniformly and quadratically to a unique solution of the forced Duffing equation involving mixed nonlinearities with periodic boundary conditions.

1 Introduction

The monotone iterative technique coupled with the method of upper and lower solutions [1-7] manifests itself as an effective and flexible mechanism that offers theoretical as well as constructive existence results in a closed set, generated by the lower and upper solutions. In general, the convergence of the sequence of approximate solutions given by the monotone iterative technique is at most linear [8-9]. To obtain a sequence of approximate solutions converging quadratically, we use the method of quasilinearization [10]. This method has been developed for a variety of problems [11-20]. In reference [21], a generalized quasilinearization technique was developed for Duffing equation with periodic boundary conditions. Duffing equation is a well known nonlinear equation of applied science which is used as a powerful tool to discuss some important practical phenomena such as periodic orbit extraction, nonuniformity caused by an infinite domain, nonlinear mechanical oscillators, etc. Another important application of Duffing equation is in the field of the prediction of diseases.

The purpose of this paper is to study an extended form of the generalized quasilinearization method for a periodic boundary value problem involving the forced Duffing equation with mixed type of nonlinearities. Precisely, we obtain a sequence of approximate solutions converging monotonically and quadratically to a unique solution of the following problem

$$-u''(x) - ku'(x) = f(x, u(x)), \quad x \in [0, \pi], \tag{1}$$

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$$u(0) = u(\pi), \qquad u'(0) = u'(\pi), \tag{2}$$

where $f \in [0,\pi] \times R \to R$ is continuous and is allowed to admit a decomposition of the form

$$f(x,u) = h(x,u) + g(x,u).$$

Here, we assume that $(h(x,u) + M_1 u^2)$ is convex for some $M_1 > 0$ and instead of requiring the convexity/concavity assumption on $g(x,u)$, we demand a less restrictive condition on $g(x,u)$ namely, $[g(x,u) + M_2 u^{\epsilon+1}]$ satisfies a nondecreasing condition for some $M_2 > 0$ and $\epsilon > 0$.

2 Preliminaries

We observe that the homogeneous periodic boundary value problem

$$-u''(x) - ku'(x) - \lambda u(x) = 0, \quad x \in [0,\pi],$$

$$u(0) = u(\pi), \qquad u'(0) = u'(\pi),$$

has only the trivial solution if and only if $\lambda \neq 4n^2$ for $k = 0$ and $\lambda \neq 0$ for $k \neq 0$ for all $n \in \{0,1,2,...\}$. Consequently, for these values of λ, the solution $u(x)$ of (1)-(2) can be written by Green's function method as

$$u(x) = \int_0^\pi G_\lambda(x,y) f(y,u(y))dy, \tag{3}$$

where $G_\lambda(x,y)$ is the Green's function given by

$$G_\lambda(x,y) = \begin{cases} \gamma_1 \left[\sinh\left(\frac{\sqrt{k^2-4\lambda}(\pi-(y-x))}{2}\right) + e^{\frac{-k}{2}\pi}\sinh\left(\frac{\sqrt{k^2-4\lambda}(y-x)}{2}\right)\right], & 0 \le x \le y, \\ \gamma_1 \left[\sinh\left(\frac{\sqrt{k^2-4\lambda}(\pi-(x-y)}{2}\right) + e^{\frac{k}{2}\pi}\sinh\left(\frac{\sqrt{k^2-4\lambda}(x-y)}{2}\right)\right], & y \le x \le \pi, \end{cases}$$

for $\lambda < \frac{k^2}{4}$ and

$$G_\lambda(x,y) = \begin{cases} \gamma_2 \left[\sin\left(\frac{\sqrt{4\lambda-k^2}(\pi-(y-x)}{2}\right) + e^{\frac{-k}{2}\pi}\sin\left(\frac{\sqrt{4\lambda-k^2}(y-x)}{2}\right)\right], & 0 \le x \le y, \\ \gamma_2 \left[\sin\left(\frac{\sqrt{4\lambda-k^2}(\pi-(x-y)}{2}\right) + e^{\frac{k}{2}\pi}\sin\left(\frac{\sqrt{4\lambda-k^2}(x-y)}{2}\right)\right], & y \le x \le \pi, \end{cases}$$

for $\lambda > \frac{k^2}{4}$, where

$$\gamma_1 = \frac{2e^{\frac{-k}{2}(x+\pi-y)}}{\sqrt{k^2-4\lambda}\left(2e^{\frac{-k}{2}\pi}\cosh\left(\frac{\sqrt{k^2-4\lambda}}{2}\pi\right) - 1 - e^{-k\pi}\right)},$$

$$\gamma_2 = \frac{-2e^{\frac{-k}{2}(x+\pi-y)}}{\sqrt{4\lambda-k^2}\left(1 + e^{-k\pi} - 2e^{\frac{-k}{2}\pi}\cos\left(\frac{\sqrt{4\lambda-k^2}}{2}\pi\right)\right)}.$$

We note that $G_\lambda(x,y) > 0$ for $\lambda < 0$ and $G_\lambda(x,y) \le 0$ for $\frac{k^2}{4} < \lambda \le \frac{k^2}{4} + 1$.

We say that $\alpha \in C^2([0,\pi])$ is a lower solution of (1)-(2) if

$$-\alpha''(x) - k\alpha'(x) \leq f(x, \alpha(x)), \quad x \in [0,\pi],$$

$$\alpha(0) = \alpha(\pi), \qquad \alpha'(0) \geq \alpha'(\pi).$$

Similarly, $\beta \in C^2([0,\pi])$ is an upper solution of (1)-(2) if

$$-\beta''(x) - k\beta'(x) \geq f(x, \beta(x)), \quad x \in [0,\pi],$$

$$\beta(0) = \beta(\pi), \qquad \beta'(0) \leq \beta'(\pi).$$

We state the following results to prove the main result. We do not provide the proof of these results as it is based on the standard arguments [1, 11, 12].

THEOREM 1. Assume that $\alpha, \beta \in C^2[0,\pi]$ are respectively lower and upper solutions of the boundary value problem (1)-(2). If $f(x,u)$ is strictly decreasing in u for each $x \in [0,\pi]$, then $\alpha(x) \leq \beta(x)$ for $x \in [0,\pi]$.

THEOREM 2. Suppose that $\alpha, \beta \in C^2([0,\pi])$ are lower and upper solutions of (1)-(2) respectively such that $\alpha(x) \leq \beta(x)$ for all $x \in [0,\pi]$. Then there exists at least one solution $u(x)$ of (1)-(2) such that $\alpha(x) \leq u(x) \leq \beta(x)$ for $x \in [0,\pi]$.

3 Main Result

THEOREM 3. Assume that

($\mathbf{A_1}$) $\alpha, \beta \in C^2([0,\pi])$ are lower and upper solutions of (1)-(2) respectively such that $\alpha(x) \leq \beta(x)$ for all $x \in [0,\pi]$;

($\mathbf{A_2}$) $h_u(x,u), h_{uu}(x,u)$ exist, are continuous with $h_{uu}(x,u) + 2M_1 \geq 0$ for all $(x,u) \in \Omega$, where $\Omega = \{(x,u) \in R^2 : x \in [0,\pi], \alpha(x) \leq u(x) \leq \beta(x)\}$;

($\mathbf{A_3}$) $g_u(x,u)$ exists and satisfy

$$\{[g_u(x,u) + (1+\epsilon)M_2 u^{\epsilon}] - [g_u(x,v) + (1+\epsilon)M_2 v^{\epsilon}]\}(u-v) \geq 0, \quad \epsilon > 0;$$

($\mathbf{A_4}$) $h_u(x,u) + g_u(x,u) < 0$ for all $x \in [0,\pi]$.

Then there exists a monotone sequence $\{\alpha_n\}$ which converges uniformly and quadratically to a unique solution of (1)-(2).

PROOF. For any $u \geq v$, it follows from (A_2) and (A_3) that

$$h(x,u) + g(x,u) \geq G(x,u,v), \quad G(x,u,u) = h(x,u) + g(x,u) \tag{4}$$

where

$$\begin{aligned} G(x,u,v) \quad = \quad & h(x,v) + [h_u(x,v) + 2M_1 v](u-v) - M_1(u^2 - v^2) \\ & + g(x,v) + [g_u(x,v) + (1+\epsilon)M_2 v^{\epsilon}](u-v) - M_2(u^{(1+\epsilon)} - v^{(1+\epsilon)}). \end{aligned} \tag{5}$$

Further, in view of (A_4), it follows that $G_u(x,u,v) < 0$ for each fixed $(x,v) \in [0,\pi] \times \mathbb{R}$. Now, we set $\alpha_0(x) = \alpha(x)$ and consider the problem

$$-u''(x) - ku'(x) = G(x, u(x), \alpha_0(x)), \quad x \in [0,\pi], \tag{6}$$

$$u(0) = u(\pi), \qquad u'(0) = u'(\pi). \tag{7}$$

In view of (A_1), (4) and (5), we have

$$-\alpha_0''(x) - k\alpha_0'(x) \le h(x,\alpha_0(x)) + g(x,\alpha_0(x)) = G(x,\alpha_0(x),\alpha_0(x)), \quad x \in [0,\pi],$$

$$\alpha_0(0) = \alpha_0(\pi), \qquad \alpha_0'(0) \ge \alpha_0'(\pi),$$

$$-\beta''(x) - k\beta'(x) \ge h(x,\beta(x)) + g(x,\beta(x)) \ge G(x,\beta(x),\alpha_0(x)), \quad x \in [0,\pi],$$

$$\beta(0) = \beta(\pi), \qquad \beta'(0) \le \beta'(\pi),$$

which imply that $\alpha_0(x)$ and $\beta(x)$ are lower and upper solutions of (6)-(7) respectively. Hence, by Theorems 1 and 2, there exists a unique solution $\alpha_1(x)$ of (6)-(7) such that $\alpha_0(x) \le \alpha_1(x) \le \beta(x)$ for $x \in [0,\pi]$.

Next, consider the problem

$$-u''(x) - ku'(x) = G(x, u(x), \alpha_1(x)), \quad x \in [0,\pi], \tag{8}$$

$$u(0) = u(\pi), \qquad u'(0) = u'(\pi). \tag{9}$$

Using (4) and (5) together with the fact that $\alpha_1(x)$ is a solution of (6)-(7), we obtain

$$\begin{aligned} & -\alpha_1''(x) - k\alpha_1'(x) = G(x,\alpha_1(x),\alpha_0(x)) \\ \le \quad & h(x,\alpha_1(x)) + g(x,\alpha_1(x)) = G(x,\alpha_1(x),\alpha_1(x)), \quad x \in [0,\pi], \\ & \qquad \alpha_1(0) = \alpha_1(\pi), \qquad \alpha_1'(0) = \alpha_1'(\pi), \\ & -\beta''(x) - k\beta'(x) \ge h(x,\beta(x)) + g(x,\beta(x)) \ge G(x,\beta(x),\alpha_1(x)), \quad x \in [0,\pi], \\ & \qquad \beta(0) = \beta(\pi), \qquad \beta'(0) \le \beta'(\pi). \end{aligned}$$

Thus it follows that $\alpha_1(x)$ and $\beta(x)$ are respectively lower and upper solutions of (8)-(9). Again, by Theorems 1 and 2, there exists a unique solution $\alpha_2(x)$ of (8)-(9) such that

$$\alpha_1(x) \le \alpha_2(x) \le \beta(x), \quad x \in [0,\pi].$$

Continuing this process successively, we obtain a monotone sequence $\{\alpha_n(x)\}$ satisfying

$$\alpha_0(x) \le \alpha_1(x) \le ... \le \alpha_n(x) \le \beta(x), \quad x \in [0,\pi],$$

where $\alpha_n(x)$ is the unique solution of the problem

$$u''(x) - ku'(x) = G(x, u(x), \alpha_{n-1}(x)), \quad x \in [0,\pi],$$

$$u(0) = u(\pi), \qquad u'(0) = u'(\pi).$$

Since the sequence $\{\alpha_n(x)\}$ is monotone, it follows that it has a pointwise limit $u(x)$.

To show that $u(x)$ is in fact a solution of (1)-(2), we note that $\alpha_n(x)$ is a solution of the following problem

$$-u''(x) - ku'(x) - \lambda u(x) = \psi_n(x), \quad x \in [0, \pi], \tag{10}$$

$$u(0) = u(\pi), \quad u'(0) = u'(\pi), \tag{11}$$

where $\psi_n(x) = G(x, \alpha_n(x), \alpha_{n-1}(x)) - \lambda\alpha_n(x)$ for every $x \in [0, \pi]$. Since $G(x, u, v)$ is continuous on Ω and $\alpha_0(x) \leq \alpha_n(x) \leq \beta(x)$ on $[0, \pi]$, it follows that $\psi_n(x)$ is bounded in $C[0, \pi]$. Thus, $\alpha_n(x)$, the solution of (10)-(11) can be written as

$$\alpha_n(x) = \int_0^\pi G_\lambda(x, y)\psi_n(y)dy. \tag{12}$$

Since $[0, \pi]$ is compact and the monotone convergence of the sequence $\{\alpha_n(x)\}$ is pointwise, it follows by the standard arguments (Arzela Ascoli convergence criterion, Dini's theorem [1, 12]) that the convergence of the sequence is uniform. Thus, taking the limit $n \to \infty$ in (12) yields

$$u(x) = \int_0^\pi G_\lambda(x, y)[h(y, u(y)) + g(y, u(y)) - \lambda u(y)]dy, \quad x \in [0, \pi].$$

Thus, $u(x)$ is a solution of (1)-(2). Now, we prove that the convergence of the sequence is quadratic. For that, setting $e_n(x) = u(x) - \alpha_n(x) \geq 0$, $n = 1, 2, 3, \ldots$ and $P(x, u(x)) = h(x, u(x)) + g(x, u(x)) + M_1u^2(x) + M_2u^{1+\epsilon}(x)$, we obtain $e_n(0) = e_n(\pi)$, $e_n'(0) = e_n'(\pi)$ and

$$\begin{aligned}
& -e_n''(x) - ke_n'(x) \\
= \; & \alpha_n''(x) + k\alpha_n'(x) - u''(x) - ku'(x) \\
= \; & -G(x, \alpha_n(x), \alpha_{n-1}(x)) + h(x, u(x)) + g(x, u(x)) \\
= \; & -[h(x, \alpha_{n-1}(x)) + \{h_u(x, \alpha_{n-1}(x)) + 2M_1\alpha_{n-1}(x)\}(\alpha_n(x) - \alpha_{n-1}(x)) \\
& -M_1(\alpha_n^2(x) - \alpha_{n-1}^2(x)) + g(x, \alpha_{n-1}(x)) \\
& +\{g_u(x, \alpha_{n-1}(x)) + (1+\epsilon)M_2\alpha_{n-1}^\epsilon(x)\}(\alpha_n(x) - \alpha_{n-1}(x)) \\
& -M_2(\alpha_n^{1+\epsilon}(x) - \alpha_{n-1}^{1+\epsilon}(x))] + h(x, u(x)) + g(x, u(x)) \\
= \; & -P(x, \alpha_{n-1}(x)) + P(x, u(x)) - M_1u^2(x) - M_2u^{1+\epsilon}(x) \\
& -P_u(x, \alpha_{n-1}(x))(\alpha_n(x) - \alpha_{n-1}(x)) + M_1\alpha_n^2(x) + M_2\alpha_n^{1+\epsilon}(x) \\
= \; & P_u(x, \xi)(u(x) - \alpha_{n-1}(x)) - M_1(u^2(x) - \alpha_n^2(x)) - M_2(u^{1+\epsilon}(x) - \alpha_n^{1+\epsilon}(x)) \\
& +P_u(x, \alpha_{n-1}(x))(u(x) - \alpha_n(x)) - P_u(x, \alpha_{n-1}(x))(u(x) - \alpha_{n-1}(x)) \\
= \; & [P_u(x, \xi) - P_u(x, \alpha_{n-1}(x))](u(x) - \alpha_{n-1}(x)) - M_1(u(x) + \alpha_n(x))(u(x) - \alpha_n(x)) \\
& -M_2(u(x) - \alpha_n(x))\eta(u(x), \alpha_n(x)) + P_u(x, \alpha_{n-1}(x))(u(x) - \alpha_n(x)) \\
= \; & [P_u(x, \xi) - P_u(x, \alpha_{n-1}(x))](u(x) - \alpha_{n-1}(x)) \\
& +[P_u(x, \alpha_{n-1}(x)) - M_1(u(x) + \alpha_n(x)) - M_2\eta(u(x), \alpha_n(x))](u(x) - \alpha_n(x))
\end{aligned}$$

$$
\begin{aligned}
&= [P_u(x,\xi) - P_u(x,\alpha_{n-1}(x))]e_{n-1}(x) + [P_u(x,\alpha_{n-1}(x)) - M_1(u(x)+\alpha_n(x)) \\
&\quad - M_2\eta(u(x),\alpha_n(x))]e_n(x) \\
&= [h_u(x,\xi) - h_u(x,\alpha_{n-1}(x)) + g_u(x,\xi) - g_u(x,\alpha_{n-1}(x)) + 2M_1(\xi - \alpha_{n-1}(x)) \\
&\quad +(1+\epsilon)M_2(\xi^\epsilon - \alpha_{n-1}^\epsilon(x))]e_{n-1}(x) + [P_u(x,\alpha_{n-1}(x)) - M_1(u(x)+\alpha_n(x)) \\
&\quad - M_2\eta(u(x),\alpha_n(x))]e_n(x) \\
&= [h_{uu}(x,\sigma)(\xi - \alpha_{n-1}(x)) + g_u(x,\xi) - g_u(x,\alpha_{n-1}(x)) + 2M_1(\xi - \alpha_{n-1}(x)) \\
&\quad +(1+\epsilon)M_2(\xi^\epsilon - \alpha_{n-1}^\epsilon(x))]e_{n-1}(x) + [P_u(x,\alpha_{n-1}(x)) - M_1(u(x)+\alpha_n(x)) \\
&\quad - M_2\eta(u(x),\alpha_n(x))]e_n(x),
\end{aligned}
\tag{13}
$$

where $\alpha_{n-1}(x) \leq \sigma \leq \xi \leq u(x)$ on $[0,\pi]$ and $\eta(a,b) > 0$ for all a, b. Substituting

$$P_u(x,\alpha_{n-1}(x)) - M_1(u(x)+\alpha_n(x)) - M_2\eta(u(x),\alpha_n(x)) = a_n(x)$$

and using the estimate

$$
\begin{aligned}
&(h_{uu}(x,\sigma) + 2M_1)(\xi - \alpha_{n-1}(x)) + g_u(x,\xi) - g_u(x,\alpha_{n-1}(x)) \\
&+(1+\epsilon)M_2(\xi^\epsilon - \alpha_{n-1}^\epsilon(x)) \leq V(\xi - \alpha_{n-1}(x)) \leq V e_{n-1}(x)
\end{aligned}
$$

in (13) gives

$$-e_n''(x) - ke_n'(x) - e_n(x)a_n(x) = Ve_{n-1}^2(x) + b_n(x), \quad x \in [0,\pi],$$

$$e_n(0) = e_n(\pi), \qquad e_n'(0) = e_n'(\pi),$$

where $b_n(x) \leq 0$ on $[0,\pi]$. Since $\lim_{n\to\infty} a_n(x) = P_u(x,u(x)) - 2M_1 u(x) - (1+\epsilon)M_2 u(x) = h_u(x,u(x)) + g_u(x,u(x))$ and $h_u(x,u(x)) + g_u(x,u(x)) < 0$, therefore, for $\lambda < 0$, there exists $n_0 \in \mathbb{N}$ such that $a_n(x) - \lambda < 0,\ x \in [0,\pi]$ for $n \geq n_0$. Therefore, the error function $e_n(x)$ satisfies the following problem

$$-e_n''(x) - ke_n'(x) - \lambda e_n(x) = (a_n(x) - \lambda)e_n(x) + Ve_{n-1}^2(x) + b_n(x), \quad x \in [0,\pi],$$

whose solution is

$$e_n(x) = \int_0^\pi G_\lambda(x,y)[(a_n(y) - \lambda)e_n(y) + Ve_{n-1}^2(y) + b_n(y)]dy.$$

Since $a_n(y) - \lambda < 0, \ \ b_n(y) \leq 0$, and $G_\lambda(x,y) > 0$ for $\lambda < 0$, therefore, it follows that

$$G_\lambda(x,y)[(a_n(y) - \lambda)e_n(y) + Ve_{n-1}^2(y) + b_n(y)] < G_\lambda(x,y)Ve_{n-1}^2(y).$$

Thus, we obtain

$$0 \leq e_n(x) \leq V\int_0^\pi G_\lambda(x,y)e_{n-1}^2(y)dy,$$

which can be expressed as

$$\|e_n\| \leq V^*\|e_{n-1}\|^2,$$

where $V^* = V \max \int_0^\pi G_\lambda(x,y)dy$ and $\|e_n\| = \max\{|e_n| : x \in [0,\pi]\}$ is the usual uniform norm. This completes the proof of the theorem.

REMARK. It is interesting to note that the generalized quasilinearization technique for the PBVP (1)-(2) [21] follows as a special case if we take $g(x,u) = 0$ in the forcing term of the Duffing equation.

EXAMPLE. Consider the boundary value problem

$$-u''(x) - ku'(x) = e^{a(1-u(x))} - \pi + |u(x)|^2, \quad a > 2, \ x \in [0,\pi], \tag{14}$$

$$u(0) = u(\pi), \quad u'(0) = u'(\pi). \tag{15}$$

Here, $h(x,u(x)) = e^{a(1-u(x))} - \pi$, $g(x,u(x)) = |u(x)|^2$. Let $\alpha(x) = -1$ and $\beta(x) = 1$ be lower and upper solutions of (14)-(15) respectively. By choosing $M_1 > 0$, $0 < M_2 \leq 1, \epsilon = 1$, it can easily be verified that the assumptions $(A_1) - (A_4)$ of Theorem 3 are satisfied. Thus, the conclusion of Theorem 3 applies to the problem (14)-(15).

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