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# Positive Solutions For A Singular Second Order Boundary Value Problem*

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**Abstract**

In this paper we obtain sufficient conditions of existence of positive solutions for a singular second order boundary value problem. Our argument is based on regularization technique, upper and lower solutions method and the Arzelá-Ascoli theorem.

## 1 Introduction

In [1], Bertsch and Ughi investigated the following BVP which arises in study of a class of degenerate parabolic equations (also see [2, 3]):

$$\begin{cases} u'' + \dfrac{N-1}{t}u' - \gamma\dfrac{|u'|^2}{u} + 1 = 0, \quad 0 < t < 1, \\ u(1) = u'(0) = 0, \end{cases} \tag{1}$$

where $N$ is a positive integer and $\gamma > 0$, and obtained one decreasing positive solution via theories of ordinary differential equation. In the very recent paper [4], the authors considered the following BVP:

$$\begin{cases} u'' + \dfrac{\lambda}{t}u' - \gamma\dfrac{|u'|^2}{u} + f(t) = 0, \quad 0 < t < 1, \\ u(1) = u'(0) = 0, \end{cases} \tag{2}$$

and proved, by the classical method of elliptic regularization, that BVP (2) has one positive solution which is not decreasing in the case: $\lambda > 0, \gamma > \frac{1+\lambda}{2}, f \in C[0,1]$ and $f > 0$ on $[0,1]$.

This paper considers the more general problem:

$$\begin{cases} u'' + \lambda\dfrac{u'}{t^m} - \gamma\dfrac{|u'|^2}{u^p} + f(t) = 0, \quad 0 < t < 1, \\ u(1) = u'(0) = 0, \end{cases} \tag{3}$$

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where $\lambda, m, \gamma, p > 0$, $f(t) \in C[0,1]$ and $f(t) > 0$ on $[0,1]$. By a solution to BVP (3) we mean a function $u \in C^2(0,1) \cap C^1[0,1]$ which is positive in (0, 1) and satisfies (3). By an argument based on the regularization technique, upper and lower solutions and the Arzelá-Ascoli theorem, we obtain sufficient conditions of existence of solutions. Our main result reads

THEOREM 1. Let $\lambda \in (0, +\infty), p \in [1, 2), m \in (0, p/(2-p)]$, and let $f \in C[0,1]$ and $f(t) > 0$ on $[0,1]$. If $\gamma > \inf_{t \geqslant 1} \mathcal{G}(t)$, where $\mathcal{G}(t) : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$\mathcal{G}(t) = \frac{p + \lambda(2-p)}{2} t^{p-1} + \frac{(2-p)^2 \max_{[0,1]} f}{4} t^{p-2},$$

then BVP (3) has at least one solution.

REMARK 1. If $p = 1$, then $\inf_{t \geqslant 1} \mathcal{G}(t) = \frac{1+\lambda}{2}$. Clearly, Theorem 1 is an extension of the existence results of [1, 4].

REMARK 2. Let $p \in (1, 2)$, and denote

$$T_0 = \frac{(2-p)^3 \max_{[0,1]} f}{2(p-1)[p + \lambda(2-p)]}, \qquad T_* = \begin{cases} T_0, & T_0 \geqslant 1, \\ 1, & T_0 < 1. \end{cases}$$

Then $\inf_{t \geqslant 1} \mathcal{G}(t) = \mathcal{G}(T_*)$. Indeed, since $\lim_{t \to 0^+} \mathcal{G}(t) = \lim_{t \to +\infty} \mathcal{G}(t) = +\infty$, $\mathcal{G}(t)$ must reach a minimum at some point $t \in (0, \infty)$ such that $\mathcal{G}'(t) = 0$, and then, solving this equation yields $t = T_0$ and hence, $\inf_{t > 0} \mathcal{G}(t) = \mathcal{G}(T_0)$. Since $\mathcal{G}'(t) \geqslant 0$ for all $t \geqslant T_0$, we see that $\inf_{t \geqslant 1} \mathcal{G}(t) = \inf_{t > 0} \mathcal{G}(t) = \mathcal{G}(T_0)$ if $T_0 \geqslant 1$, and $\inf_{t \geqslant 1} \mathcal{G}(t) = \mathcal{G}(1)$ if $T_0 < 1$.

## 2 Proof of Theorem 1

Let $\epsilon \in (0,1)$, and define $H_\epsilon(t, v, \xi) : (0,1) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$H_\epsilon(t, v, \xi) = -\lambda \frac{\xi}{(t + \epsilon^{1/\alpha})^m} + \gamma \frac{|\xi|^2}{[I_\epsilon(v)]^p} - f(t),$$

where $\alpha = \frac{2}{2-p}$, and $I_\epsilon(v) = v + \epsilon^2$ if $v \geqslant 0$, $I_\epsilon(v) = \epsilon^2$ if $v < 0$. We have

$$\begin{aligned} |H_\epsilon(t, v, \xi)| &\leqslant \frac{\lambda}{\epsilon^{m/\alpha}} |\xi| + \gamma \frac{|\xi|^2}{\epsilon^{2p}} + \max_{[0,1]} f \\ &\leqslant \frac{\lambda}{\epsilon^{m/\alpha}} (1 + |\xi|^2) + \frac{\gamma}{\epsilon^{2p}} |\xi|^2 + \max_{[0,1]} f \\ &\leqslant \Big( \frac{\lambda}{\epsilon^{m/\alpha}} + \frac{\gamma}{\epsilon^{2p}} + \max_{[0,1]} f \Big) \mathcal{H}(|\xi|) \end{aligned} \tag{4}$$

for all $(t, v, \xi) \in (0,1) \times \mathbb{R} \times \mathbb{R}$, where $\mathcal{H}(s) = 1 + s^2$ for $s \geqslant 0$. Define operator $L_\epsilon : C^2(0,1) \to C(0,1)$ by

$$(L_\epsilon u)(t) = -u'' + H_\epsilon(t, u, u'), \quad 0 < t < 1.$$

Consider the problem:

$$\begin{cases} (L_\epsilon u)(t) = 0, \quad 0 < t < 1, \\ u(1) = u(0) = 0. \end{cases} \tag{5}$$

We call $u$ an upper solution (lower solution) of problem (5) if $L_\epsilon u \geqslant (\leqslant) 0$ in $(0,1)$, and $u(t) \geqslant (\leqslant) 0$ for $t = 0, 1$.

We will apply the upper and lower solutions method (see [5, pp.153, Theorem 2.5.4] or [6, Theorem 1 and Remark 2.4]) to obtain one positive solution of problem (5). Note that $\int_0^{+\infty} \frac{s}{\mathcal{H}(s)} ds = +\infty$. Then it suffices to find a lower solution and an upper solution to obtain a solution.

LEMMA 1. Let $U = C_1 W^\alpha$ with $\alpha = \frac{2}{2-p}$, where $W(t) = t(1-t)$ and $C_1 \in (0,1)$ such that $2C_1\alpha + C_1\alpha\lambda 2^{\alpha-1-m} + \gamma C_1^{2-p}\alpha^2 \leqslant \min\limits_{[0,1]} f(t)$. Then $U$ is a lower solution of problem (5).

PROOF. Note that $W'' = -2, W \leqslant t$ and $|W'| \leqslant 1$ on $[0,1]$. Since $U > 0$ in $(0,1)$, some calculations give by noticing $\alpha \geqslant 1 + m$

$$\begin{aligned} L_\epsilon U &= -U'' - \lambda \frac{U'}{(t + \epsilon^{1/\alpha})^m} + \gamma \frac{|U'|^2}{(U + \epsilon^2)^p} - f(t) \\ &\leqslant -U'' - \lambda \frac{U'}{(t + \epsilon^{1/\alpha})^m} + \gamma \frac{|U'|^2}{U^p} - f(t) \\ &= 2C_1\alpha W^{\alpha-1} - C_1\alpha(\alpha - 1)W^{\alpha-2}|W'|^2 \\ &\quad + C_1\alpha\lambda \frac{W^{\alpha-1}W'}{(t + \epsilon^{1/\alpha})^m} + \gamma C_1^{2-p}\alpha^2|W'|^2 - f(t) \\ &\leqslant 2C_1\alpha W^{\alpha-1} + C_1\alpha\lambda \frac{W^{\alpha-1}W'}{(t + \epsilon^{1/\alpha})^m} + \gamma C_1^{2-p}\alpha^2|W'|^2 - f(t) \\ &\leqslant 2C_1\alpha + C_1\alpha\lambda(t + \epsilon^{1/\alpha})^{\alpha-1-m} + \gamma C_1^{2-p}\alpha^2 - f(t) \\ &\leqslant 2C_1\alpha + C_1\alpha\lambda 2^{\alpha-1-m} + \gamma C_1^{2-p}\alpha^2 - \min_{[0,1]} f(t) \\ &\leqslant 0, \quad 0 < t < 1. \end{aligned}$$

Thus, $U$ is a lower solution of problem (5). The lemma follows.

Let $\inf\limits_{s\geqslant 1} H(s) \equiv \delta$. Then it follows from the definition of infimum and $\gamma > \delta$ that for $\delta_0 = \frac{\gamma-\delta}{2} > 0$, there exists $C_* \geqslant 1$, such that $H(C_*) < \delta + \delta_0 < \gamma$.

LEMMA 2. There exists a positive constant $\epsilon_0 \in (0,1)$, such that for any $\epsilon \in (0, \epsilon_0)$, $V_\epsilon = C_*(t + \epsilon^{\frac{1}{\alpha}})^\alpha$ is an upper solution of problem (5).

PROOF. Noticing $\alpha \geqslant 2$ and $1+m \leqslant \alpha$, we have

$$\begin{aligned}
L_\epsilon V_\epsilon &= -V_\epsilon'' - \lambda \frac{V_\epsilon'}{(t+\epsilon^{1/\alpha})^m} + \gamma \frac{|V_\epsilon'|^2}{(V_\epsilon+\epsilon^2)^p} - f(t) \\
&= -C_*\alpha(\alpha-1)(t+\epsilon^{1/\alpha})^{\alpha-2} - \lambda\alpha C_*(t+\epsilon^{1/\alpha})^{\alpha-1-m} \\
&\quad + \gamma C_*^{2-p}\alpha^2[1+\epsilon^2 C_*^{-1}(t+\epsilon^{1/\alpha})^{-\alpha}]^{-p} - f(t) \\
&\geqslant -C_*\alpha(\alpha-1)[1+\epsilon^{1/\alpha}]^{\alpha-2} - \lambda\alpha C_*[1+\epsilon^{1/\alpha}]^{\alpha-1-m} \\
&\quad + \gamma C_*^{2-p}\alpha^2[1+\epsilon C_*^{-1}]^{-p} - \max_{[0,1]} f(s) \\
&= \gamma C_*^{2-p}\alpha^2 - C_*\alpha(\alpha-1) - \lambda\alpha C_* - \max_{[0,1]} f(s) + e_\epsilon \\
&= C_*^{2-p}\alpha^2(\gamma - \mathcal{G}(C_*)) + e_\epsilon, \quad 0<t<1,
\end{aligned}$$

where $e_\epsilon = C_*\alpha(\alpha-1)[1-(1+\epsilon^{1/\alpha})^{\alpha-2}] + \lambda\alpha C_*[1-(1+\epsilon^{1/\alpha})^{\alpha-1-m}] + [1+\epsilon C_*^{-1}]^{-p} - 1$. Clearly, $e_\epsilon \to 0, \ \ (\epsilon \to 0)$. Since $\gamma > \mathcal{G}(C_*)$, there exists $\epsilon_0 \in (0,1)$ such that

$$C_*^{2-p}\alpha^2(\gamma - \mathcal{G}(C_*)) + e_\epsilon \geqslant 0.$$

This shows that for any $\epsilon \in (0,\epsilon_0)$, $L_\epsilon V_\epsilon \geqslant 0, 0<t<1$. The lemma follows.

According to [5, pp.153, Theorem 2.5.4] or [6, Theorem 1 and Remark 2.4], for any fixed $\epsilon \in (0,\epsilon_0)$, problem (5) has a solution $u_\epsilon \in C^1[0,1]$ satisfying $u_\epsilon' \in C^1(0,1)$ and

$$V_\epsilon \geqslant u_\epsilon \geqslant U > 0, \quad t \in (0,1). \tag{6}$$

Hence $u_\epsilon$ satisfies

$$u_\epsilon'' + \lambda \frac{u_\epsilon'}{(t+\epsilon^{1/\alpha})^m} - \gamma \frac{|u_\epsilon'|^2}{(u_\epsilon+\epsilon^2)^p} + f(t) = 0, \quad 0<t<1. \tag{7}$$

LEMMA 3. There exists a positive constant $C_2$ independent of $\epsilon$, such that for all $\epsilon \in (0,\epsilon_0)$

$$|u_\epsilon'(t)| \leqslant C_2, \quad t \in [0,1]. \tag{8}$$

PROOF. It follows from $u_\epsilon(1) = u_\epsilon(0) = 0$ and $u_\epsilon \geqslant 0$ for all $t \in [0,1]$ that

$$u_\epsilon'(0) \geqslant 0 \geqslant u_\epsilon'(1). \tag{9}$$

Integrating (7) over (0, 1) and integrating by parts give

$$\begin{aligned}
& u_\epsilon'(t)\Big|_0^1 + \frac{\lambda u_\epsilon(t)}{(t+\epsilon^{1/\alpha})^m}\Big|_0^1 + m\lambda\int_0^1 \frac{u_\epsilon}{(t+\epsilon^{1/\alpha})^{1+m}}dt \\
& - \gamma\int_0^1 \frac{|u_\epsilon'|^2}{(u_\epsilon+\epsilon^2)^p}dt + \int_0^1 f(t)dt = 0,
\end{aligned}$$

and then, we obtain by (9)

$$\gamma\int_0^1 \frac{|u_\epsilon'|^2}{(u_\epsilon+\epsilon^2)^p}dt \leqslant \frac{\lambda u_\epsilon(t)}{(t+\epsilon^{1/\alpha})^m}\Big|_0^1 + m\lambda\int_0^1 \frac{u_\epsilon}{(t+\epsilon^{1/\alpha})^{1+m}}dt + \int_0^1 f(t)dt.$$

Since $m \leqslant \frac{p}{2-p}$, $1+m \leqslant \alpha = \frac{2}{2-p}$. From (6), it is easy to see that $\frac{\lambda u_\epsilon(t)}{(t+\epsilon^{1/\alpha})^m}\Big|_0^1 + m\lambda\int_0^1 \frac{u_\epsilon}{(t+\epsilon^{1/\alpha})^{1+m}}dt$ is uniformly bounded and hence, there exists a positive constant $C_3$ independent of $\epsilon$, such that

$$\int_0^1 \frac{|u_\epsilon'|^2}{(u_\epsilon+\epsilon^2)^p}dt \leqslant C_3. \tag{10}$$

By the inequality: $a \leqslant a^2+1 \quad (a \in \mathbb{R})$, we obtain

$$\frac{|u_\epsilon'|}{(t+\epsilon^{1/\alpha})^m} \leqslant \frac{|u_\epsilon'|^2}{(t+\epsilon^{1/\alpha})^{2m}} + 1, \quad t \in [0,1]. \tag{11}$$

By (6), we have $u_\epsilon + \epsilon^2 \leqslant 2C_*(t+\epsilon^{1/\alpha})^\alpha, \quad t \in [0,1]$. Noticing $\alpha p \geqslant 2m$, we see that there exists a positive constant $C_4$ independent of $\epsilon$, such that

$$(u_\epsilon+\epsilon^2)^p \leqslant C_4(t+\epsilon^{1/\alpha})^{2m}, \quad t \in [0,1].$$

Combining this and (11) we obtain

$$\frac{|u_\epsilon'|}{(t+\epsilon^{1/\alpha})^m} \leqslant C_4\frac{|u_\epsilon'|^2}{(u_\epsilon+\epsilon^2)^p} + 1, \quad t \in [0,1],$$

which and (10) imply that

$$\int_0^1 \frac{|u_\epsilon'|}{(t+\epsilon^{1/\alpha})^m}dt \leqslant C_3C_4 + 1 \equiv C_5. \tag{12}$$

On the other hand, integrating (7) over $(t_1, t_2)$, we have

$$u_\epsilon'(t)\Big|_{t_1}^{t_2} = -\lambda\int_{t_1}^{t_2} \frac{u_\epsilon'}{(t+\epsilon^{1/\alpha})^m}dt + \gamma\int_{t_1}^{t_2} \frac{|u_\epsilon'|^2}{(u_\epsilon+\epsilon^2)^p}dt - \int_{t_1}^{t_2} f(t)dt.$$

Combining this with (10) and (12) we obtain for all $\epsilon \in (0, \epsilon_0)$

$$|u_\epsilon'(t_2) - u_\epsilon'(t_1)| \leqslant C_6, \quad \forall t_1, t_2 \in [0,1], \tag{13}$$

where $C_6 = \lambda C_5 + \gamma C_3 + \int_0^1 f(t)dt$. Noticing $u_\epsilon(1) = u_\epsilon(0) = 0$ and using the mean value theorem, there exists $t_\epsilon \in (0,1)$, such that $u_\epsilon'(t_\epsilon) = 0$. Then taking $t_1 = t_\epsilon$ in (13), we obtain the desired result.

By (6) and (8), we derive from (7) that there exists for any $\delta \in (0, 1/2)$ a positive constant $C_\delta$ independent of $\epsilon$, such that for all $\epsilon \in (0, \epsilon_0)$

$$|u_\epsilon''(t)| \leqslant C_\delta, \quad \delta \leqslant t \leqslant 1-\delta.$$

From this and (8) and using Arzelá-Ascoli theorem, there exist a subsequence of $\{u_\epsilon\}$, still denoted by $\{u_\epsilon\}$, and a function $u \in C^1(0,1) \cap C[0,1]$ such that, as $\epsilon \to 0$,

$$\begin{aligned} &u_\epsilon \to u, \quad \text{uniformly in} \quad C[0,1], \\ &u_\epsilon \to u, \quad \text{uniformly in} \quad C^1[\delta, 1-\delta], \end{aligned}$$

and hence, by $u_\epsilon(1) = u_\epsilon(0) = 0$ and (6), $u$ satisfies $u(1) = u(0) = 0$, $C_* t^\alpha \geqslant u(t) \geqslant C_1[t(1-t)]^\alpha$ for all $t \in [0,1]$, therefore $u(t) > 0$ for all $t \in (0,1)$, and $u'(0) = \lim_{t\to 0} \frac{u(t)}{t} = 0$. Then $u$ satisfies the boundary conditions in (3).

Below, we show that $u$ satisfies the equation in (3). Integrating (7) over $[t_0, t]$ yields

$$u'_\epsilon(t) = \gamma \int_{t_0}^t \frac{|u'_\epsilon|^2}{(u_\epsilon + \epsilon^2)^p} ds - \lambda \int_{t_0}^t \frac{u'_\epsilon}{(s + \epsilon^{1/\alpha})^m} ds - \int_{t_0}^t f(s) ds + u'_\epsilon(t_0),$$

and letting $\epsilon \to 0$ and using Lebesgue dominated convergence theorem, we have

$$u'(t) = \gamma \int_{t_0}^t \frac{|u'|^2}{u^p} ds - \lambda \int_{t_0}^t \frac{u'}{s^m} ds - \int_{t_0}^t f(s) ds + u'(t_0), \quad 0 < t < 1. \tag{14}$$

From this, we see that $u \in C^2(0,1)$ and satisfies the equation in (3).

It remains to show that $u'$ is continuous at $t = 0$ and $t = 1$. Letting $\epsilon \to 0$ in (10) and (12) and using Fatou's Lemma, we have

$$\int_0^1 \frac{|u'|^2}{u^p} dt \leqslant C_3, \qquad \int_0^1 \frac{|u'|}{t^m} dt \leqslant C_5,$$

which show that $\frac{|u'|^2}{u^p}, \frac{|u'|}{t^m} \in L^1[0,1]$. By the absolute continuity of integral, we see from (14) that $u' \in C[0,1]$. Theorem 1 is proved.

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