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Dynamics Of A Rational Difference Equation Of Higher Order*

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Abstract

We investigate the global behavior of nonnegative solutions of the difference equation

$$x_{n+1} = \frac{\delta x_{n-m} + x_{n-k}}{A + x_{n-k}}, n = 0, 1, 2, ...,$$

with $\delta, A \in (0, \infty), x_{-k}, ..., x_0 \in [0, \infty)$ and $0 \le m < k$. In particular we present necessary and sufficient conditions for the existence of nonnegative prime period-two solutions. We show that the two cases $\delta < A$ and $\delta > A$ give rise to different invariant intervals. We also find regions of parameters δ and A where the equilibrium points are globally asymptotically stable. Our results extend some recent results.

1 Introduction

We investigate the global behavior of nonnegative solutions of the higher-order difference equation

$$x_{n+1} = \frac{\delta x_{n-m} + x_{n-k}}{A + x_{n-k}}, n = 0, 1, 2, ..., \tag{1}$$

with parameters $\delta, A \in (0, \infty)$, initial conditions $x_{-k}, ..., x_0 \in [0, \infty)$ and $0 \le m < k$.

Some special cases of (1) have been studied intensively. If $m = 0$, the investigation of the global behavior of all positive solutions of (1) is proposed as an open problem by Kulenvić and Ladas in [8]. Motivated by this open problem, Li and Sun [12] studied the global character of positive solutions of (1). The special case where $m = 0$ and $k = 1$ in (1) is studied in [9]. When $m = 1, k = 2$, (1) is investigated in [5], and when $m = 2, k = 3$, (1) is studied in [1]. For other related results of nonlinear rational difference equations we refer to [2-4, 6,7, 10,11, 13,14] and the references therein.

The purpose of this paper is to investigate the periodic character, invariant intervals and global asymptotic stability of the equilibrium points. Our results extend some known results (see Remarks 2.1, 3.1, 4.1 and 4.2).

The paper is organized as follows. In Section 2 we give the necessary and sufficient conditions for the existence of nonnegative prime period-two solutions of (1). In Section

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3 we show that the two cases $\delta < A$ and $\delta > A$ give rise to different invariant intervals. Finally, in Section 4 we find regions of parameters δ and A where the equilibrium points are globally asymptotically stable.

Now we present some definitions and known results which will be useful in our investigation of the behavior of solutions of (1).

Let I be some interval of real numbers, and let $f : I \times I \to I$ be a continuously differentiable function. Consider the difference equation

$$x_{n+1} = f(x_{n-m}, x_{n-k}), n = 0, 1, 2, ..., \tag{2}$$

with m, k and the initial conditions the same as those in (1).

By a *solution* of (2) we mean a sequence $\{x_n\}$ which is defined for $n \geq -k$ and satisfies (2) for $n \geq 0$. A solution $\{x_n\}$ of (2) is said to be *prime period-two* if $x_n = x_{n+2}$ and $x_n \neq x_{n+1}$ for $n \geq -k$. An interval $J \subset I$ is called an *invariant interval* for (2) if $x_{-k}, ..., x_0 \in J$ implies that $x_n \in J$ for all $n > 0$. A point $\bar{x}$ is called an *equilibrium point* of (2) if $\bar{x} = f(\bar{x}, \bar{x})$.

DEFINITION 1.1. (a) The equilibrium point $\bar{x}$ of (2) is called *stable* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, ..., x_0 \in I$ with $|x_i - \bar{x}| < \delta, i = -k, ..., 0$, we have $|x_n - \bar{x}| < \varepsilon$ for all $n > 0$.

(b) The equilibrium point $\bar{x}$ of (2) is called *locally asymptotically stable* if it is stable and if there exists $c > 0$ such that for all $x_{-k}, ..., x_0 \in I$ with $|x_i - \bar{x}| < c, i = -k, ..., 0$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

(c) The equilibrium point $\bar{x}$ of (2) is called a *global attractor* if, for every $x_{-k}, ..., x_0 \in I$, we have $\lim_{n\to\infty} x_n = \bar{x}$.

(d) The equilibrium point $\bar{x}$ of (2) is called *globally asymptotically stable* if it is stable and is a global attractor.

The *linearized equation* of (2) about an equilibrium point $\bar{x}$ of (2) is the linear difference equation

$$x_{n+1} = Px_{n-m} + Qx_{n-k}, \tag{3}$$

where

$$P = \frac{\partial f}{\partial x}(\bar{x}, \bar{x}), Q = \frac{\partial f}{\partial y}(\bar{x}, \bar{x}).$$

The *characteristic equation* of (3) is the equation

$$\lambda^{k+1} - P\lambda^{k-m} - Q = 0. \tag{4}$$

LEMMA 1.1. (Linearized stability) *If all the roots of (4) lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\bar{x}$ of (2) is locally asymptotically stable.*

2 Prime Period-Two Solution

For the existence of prime period-two solutions of (1) we have the following result.

THEOREM 2.1. (a) If m is even, (1) has no nonnegative prime period-two solution.

(b) If m is odd and k is even, (1) has a nonnegative prime period-two solution if and only if $\delta = A+1$. Furthermore, if $\delta = A+1$, then $..., \phi, \psi, \phi, \psi, ...$ is a nonnegative prime period-two solution of (1) if and only if $\phi > 1, \phi \neq 2$ and $\psi = \frac{\phi}{\phi-1}$.

(c) If m, k are odd, (1) has a nonnegative prime period-two solution if and only if $\delta + 1 > A$. Furthermore, if $\delta + 1 > A$, (1) has only two nonnegative prime period-two solutions $0, \alpha, 0, \alpha, ...$ and $\alpha, 0, \alpha, 0, ...$ with $\alpha = \delta + 1 - A$.

REMARK 2.1. If $m = 0$, statement (a) here is the same as Theorem 2.2 in [12], and if $m = 1$, $k = 2$, statement (b) is the same as Lemma 3.1 in [5] .

Proof of Theorem 2.1. (a) Assume to the contrary that (1) has a nonnegative prime period-two solution $\{x_n\}$. There are two cases to be considered.

Case 1. Suppose k is even, then $x_{n-m} = x_{n-k} = x_n$ for $n \geq 0$. It follows from (1) that

$$\begin{cases} x_1 = \frac{\delta x_{-m}+x_{-k}}{A+x_{-k}} = \frac{\delta x_0+x_0}{A+x_0}, \\ x_0 = x_2 = \frac{\delta x_{1-m}+x_{1-k}}{A+x_{1-k}} = \frac{\delta x_1+x_1}{A+x_1}, \end{cases}$$

which leads to $(A+\delta+1)(x_1 - x_0) = 0$, and then $x_1 = x_0$ since $A+\delta+1 > 0$. This contradicts the fact that $\{x_n\}$ is prime period-two.

Case 2. Suppose k is odd, then $x_{n+1} = x_{n-k}$ and $x_n = x_{n-m}$ for $n \geq 0$. By (1) we have

$$\begin{cases} x_1 = \frac{\delta x_{-m}+x_{-k}}{A+x_{-k}} = \frac{\delta x_0+x_1}{A+x_1}, \\ x_0 = x_2 = \frac{\delta x_{1-m}+x_{1-k}}{A+x_{1-k}} = \frac{\delta x_1+x_0}{A+x_0}, \end{cases} \tag{5}$$

which implies that

$$x_1 + x_0 = 1 - \delta - A. \tag{6}$$

By (5) and (6), it is easy to see that x_1 and x_0 are the two solutions of equation

$$x^2 + x(\delta + A - 1) - \delta(1 - \delta - A) = 0.$$

Then

$$x_1 x_0 (x_1 + x_0) = -\delta(1 - \delta - A)^2 \leq 0.$$

So $1-\delta-A = 0$ since $x_0, x_1 \in [0, \infty)$ and $\delta > 0$, and by (6) we have $x_0 + x_1 = 0$, which yields that $x_0 = x_1 = 0$. This also contradicts the fact that $\{x_n\}$ is prime period-two.

(b) Suppose that (1) has a nonnegative prime period-two solution $\{x_n\}$. Then $x_{n+1} = x_{n-m}$ and $x_n = x_{n-k}$ for $n \geq 0$ since m is odd and k is even, and it follows from (1) that

$$\begin{cases} x_1 = \frac{\delta x_{-m}+x_{-k}}{A+x_{-k}} = \frac{\delta x_1+x_0}{A+x_0}, \\ x_0 = x_2 = \frac{\delta x_{1-m}+x_{1-k}}{A+x_{1-k}} = \frac{\delta x_0+x_1}{A+x_1}, \end{cases}$$

which yields that $(A - \delta + 1)(x_1 - x_0) = 0$, and then $\delta = A + 1$ since $x_1 \neq x_0$.

On the other hand, if $\delta = A + 1$, let $x_{2i} = 3, x_{2i+1} = 3/2$ for $i \geq -k/2$, it is easy to verify that $\{x_n\}$ is a solution of (1), i.e., (1) has a nonnegative prime period-two solution.

Furthermore, suppose $\delta = A + 1$. It is easy to see that $..., \phi, \psi, \phi, \psi, ...$ is a nonnegative prime period-two solution of (1) if and only if

$$\psi = \frac{\delta\psi + \phi}{\delta - 1 + \phi}, \phi = \frac{\delta\phi + \psi}{\delta - 1 + \psi} \text{ with } \phi, \psi \geq 0, \phi \neq \psi, \tag{7}$$

and we can also get easily that (7) holds if and only if

$$\psi + \phi = \psi\phi \text{ with } \phi, \psi \geq 0, \phi \neq \psi. \tag{8}$$

Clearly, (8) is equivalent to that $\phi > 1, \phi \neq 2$ and $\psi = \frac{\phi}{\phi - 1}$. Therefore (b) is true.

(c) Suppose $\{x_n\}$ is a nonnegative prime period-two solution of (1). Then $x_{n+1} = x_{n-m} = x_{n-k}$ for $n \geq 0$ since m, k are odd, and we have

$$x_{n+1} = \frac{\delta x_{n-m} + x_{n-k}}{A + x_{n-k}} = \frac{\delta x_{n+1} + x_{n+1}}{A + x_{n+1}}$$

for $n \geq 0$, which implies that $x_{n+1} = \delta + 1 - A$ or $x_{n+1} = 0$ for $n \geq 0$. Hence $\delta + 1 - A > 0$ since $\{x_n\}$ is nonnegative and $x_n \neq x_{n+1}$ for $x \geq -k$. Furthermore, Let $\alpha = \delta + 1 - A > 0$, it is easily to see that $0, \alpha, 0, \alpha, ...$ and $\alpha, 0, \alpha, 0, ...$ are the only two nonnegative prime period-two solutions of (1). So statement (c) holds. This completes the proof.

3 Invariant Interval

In this section, we investigate the invariant intervals for (1).

THEOREM 3.1. Let $\{x_n\}$ be a nonnegative solution of (1). Then the following statements are true.

(a) Suppose $\delta \leq A$. If $x_{-m}, ..., x_0 \in [0, A/\delta]$, then $x_n \in [0, A/\delta]$ for $n > 0$.

(b) Suppose $\delta > A$. If $x_{-m}, ..., x_0 \in [A/\delta, \infty)$, then $x_n \in [A/\delta, \infty)$ for $n > 0$.

PROOF. (a) By (1) we have

$$x_1 = \frac{\delta x_{-m} + x_{-k}}{A + x_{-k}} \leq \frac{\delta \cdot \frac{A}{\delta} + x_{-k}}{A + x_{-k}} = 1 \leq \frac{A}{\delta}.$$

Assume that $x_i \in [0, A/\delta]$ for $1 \leq i \leq n$, we prove that $x_{n+1} \in [0, A/\delta]$. In fact, by (1) we have

$$x_{n+1} = \frac{\delta x_{n-m} + x_{n-k}}{A + x_{n-k}} \leq \frac{\delta \cdot \frac{A}{\delta} + x_{n-k}}{A + x_{n-k}} = 1 \leq \frac{A}{\delta}.$$

Therefore, $x_n \in [0, A/\delta]$ for $n > 0$.

(b) By an argument similar to the proof of (a), we can prove that statement (b) is true, and we omit the details.

If we have no the initial conditions $x_{-m}, ..., x_0 \in [0, A/\delta]$ (resp. $x_{-m}, ..., x_0 \in [A/\delta, \infty)$), the following theorem shows that we can also get that $x_n \in [0, A/\delta)$ (resp. $x_n \in (A/\delta, \infty)$) for n sufficiently large provided $\delta < A$ (resp. $\delta > A$).

THEOREM 3.2. Let $\{x_n\}$ be a nonnegative solution of (1). Then the following statements are true.

(a) If $\delta < A$, there exists N such that $x_n \in [0, A/\delta)$ for $n \geq N$.

(b) If $\delta > A$ and $x_i > 0$ for $-m \leq i \leq 0$, there exists N such that $x_n \in (A/\delta, \infty)$ for $n \geq N$.

REMARK 3.1. Theorem 3.1 and 3.2 extend Lemma 2.1 in [1], Theorem 2.3 and Theorem 2.5 in [5], Theorem 2.2 in [9] and Theorem 3.2 in [12].

Proof of Theorem 3.2. (a) Suppose on the contrary that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \geq A/\delta$ and $n_i \to \infty$. Let $M = \max\{x_{-m}, ..., x_0\}$. Then there exists a positive integer p such that $M \leq (A/\delta)^p$ since $\delta < A$, and we have $n_I \geq p(m+1)$ for some positive integer I since $n_i \to \infty$. So $n_I = p'(m+1) + q$ for some $p' \geq p$ and $-m \leq q \leq 0$.

On the other hand, if $x_{n+1} \geq A/\delta$, by (1) we can get easily that

$$x_{n-m} = \frac{A}{\delta}x_{n+1} + \frac{1}{\delta}x_{n-k}(x_{n+1} - 1) \geq \frac{A}{\delta}x_{n+1}$$

for $n \geq 0$. So we have

$$x_q \geq \left(\frac{A}{\delta}\right)^{p'} x_{q+p'(m+1)} = \left(\frac{A}{\delta}\right)^{p'} x_{n_I} \geq \left(\frac{A}{\delta}\right)^{p'+1} > \left(\frac{A}{\delta}\right)^{p} \geq M,$$

which contradicts the definition of M.

(b) Noticing that $\min\{x_{-m}, ..., x_0\} > 0$, we can prove (b) in a way similar to the proof of (a), and we omit the details. The proof is complete.

4 Global Asymptotic Stability

It is obvious that (1) has only two equilibrium points 0 and $\delta + 1 - A$. In this section, we study the global asymptotic stability of the equilibrium points. Let us start with a lemma.

LEMMA 4.1. Let $f(x, y) = (\delta x + y)/(A + y)$. Then f is increasing in x if $y \in [0, \infty)$, increasing in y if $x \in [0, A/\delta]$ and decreasing in y if $x \in [A/\delta, \infty)$.

PROOF. This follows directly from the fact that

$$\frac{\partial f(x, y)}{\partial x} = \frac{\delta}{A + y} \text{ and } \frac{\partial f(x, y)}{\partial y} = \frac{A - \delta x}{(A + y)^2}.$$

We first give a result of global asymptotic stability of the zero equilibrium of (1).

THEOREM 4.1. If $\delta \leq A - 1$, the zero equilibrium of (1) is globally asymptotically stable.

REMARK 4.1. Theorem 4.1 extends Theorem 2.1 in [1], Theorem 4.2 in [5], Theorem 2.1 (a) in [9] and the first part of Theorem 2.1 (a) in [12].

Proof of Theorem 4.1. The characteristic equation of the linearized equation of (1) about the zero equilibrium is

$$\lambda^{k+1} - \frac{\delta}{A}\lambda^{k-m} - \frac{1}{A} = 0. \tag{9}$$

Let $f(\lambda) = \lambda^{k+1}$ and $g(\lambda) = (\delta/A)\lambda^{k-m} + 1/A$. If $\delta < A - 1$, we have $|g(\lambda)| \leq |\delta/A + 1/A| < 1 = |f(\lambda)|$ for $|\lambda| = 1$. Then by Rouche's Theorem the function $f(\lambda)$

and $f(\lambda) - g(\lambda)$ have the same number of zero points in the open unit disk $|\lambda| < 1$. Therefore, all the roots of (9) satisfies $|\lambda| < 1$, and it follows from Lemma 1.1 that the zero equilibrium is locally asymptotically stable.

Let $\{x_n\}$ be any nonnegative solution of (1). If $\delta = A - 1$, it is easy to get from (1) that $x_{n+1} < \max(x_{n-k}, x_{n-m})$ provided that $x_{n-k} > 0$ or $x_{n-m} > 0$ for $n \geq 0$. From this it is easy to see that the zero equilibrium is stable.

On the other hand, by Theorem 3.2 (a), there exists N such that $x_n < A/\delta$ for $n \geq N$ since $\delta \leq A - 1$. Let $S = \limsup_{n\to\infty} x_n$, then $0 \leq S \leq A/\delta$, and it is easy to get from (1) and Lemma 4.1 that

$$S \leq \frac{(\delta+1)S}{A+S},$$

which implies that $S = 0$ since $S \geq 0$ and $\delta \leq A - 1$. Hence $\lim_{n\to\infty} x_n = 0$, i.e., zero equilibrium is a global attractor of all nonnegative solutions of (1). This completes the proof.

Finally, we investigate the global asymptotic stability of the positive equilibrium $\delta + 1 - A$ of (1) provided that $A - 1 < \delta$. We note that by the global asymptotic stability of the positive equilibrium we mean that the positive equilibrium is stable and is a global attractor of all positive solutions of (1).

THEOREM 4.2. If $0 < |\delta - A| < 1$, the positive equilibrium point $\delta + 1 - A$ of (1) is globally asymptotically stable.

REMARK 4.2. Theorem 4.2 extends Theorem 3.1 for the case $0 < |\delta - A| < 1$ in [1], Theorem 6.3 and Theorem 8.3 in [5], Theorem 5.1 for the case $A < \delta < A + 1$ and Theorem 3.2 in [9], and Theorem 3.3 and Corollary 3.1 in [12].

Proof of Theorem 4.2. The characteristic equation of the linearized equation of (1) about the positive equilibrium $\delta + 1 - A$ is

$$\lambda^{k+1} - \frac{\delta}{1+\delta}\lambda^{k-m} + \frac{\delta - A}{1+\delta} = 0. \tag{10}$$

Let

$$f(\lambda) = \lambda^{k+1} \text{ and } g(\lambda) = \frac{\delta}{1+\delta}\lambda^{k-m} - \frac{\delta - A}{1+\delta}.$$

Since $0 < |\delta - A| < 1$, for $|\lambda| = 1$ we have

$$|g(\lambda)| \leq \left|\frac{\delta}{1+\delta}\right| + \left|\frac{\delta - A}{1+\delta}\right| < 1 = |f(\lambda)|.$$

Then by Rouche's Theorem $f(\lambda)$ and $f(\lambda) - g(\lambda)$ have the same number of zero points in the open unit disk $|\lambda| < 1$. Hence all the roots of (10) satisfies $|\lambda| < 1$, and it follows from Lemma 1.1 that the equilibrium $\delta + 1 - A$ is locally asymptotically stable.

Now it is sufficient to prove the global attractivity of $\delta + 1 - A$. Let $\{x_n\}$ be any positive solution of (1). There are two cases to be considered.

Case 1. $-1 < \delta - A < 0$.

It follows from Theorem 3.2 (a) that there exists N such that $x_n < A/\delta$ for $n \geq N$. Let $a = \min\{x_N, x_{N+1}, \ldots, x_{N+k}, \delta + 1 - A\}$. Then $a > 0$ since $\{x_n\}$ is positive. We assert that

$$x_n \geq a \text{ for all } n \geq N. \tag{11}$$

If $n \geq N + k + 1$, there exists some positive integer p such that $n = p(m+1) + q$ with $N + k - m \leq q \leq N + k$, and we prove (11) by induction in p. If $p = 1$, noticing that $N \leq m + q - k \leq N + m$, by Lemma 4.1 we have

$$x_{m+1+q} = \frac{\delta x_q + x_{m+q-k}}{A + x_{m+q-k}} \geq \frac{(\delta+1)a}{A+a} \geq a,$$

i.e., (11) holds for $p = 1$. If (11) holds for $p \leq r$ (i.e., for $n \leq r(m+1) + q$ with $N + k - m \leq q \leq N + k$), by Lemma 4.1 we get

$$x_{(r+1)(m+1)+q} = \frac{\delta x_{r(m+1)+q} + x_{r(m+1)+q+m-k}}{A + x_{r(m+1)+q+m-k}} \geq \frac{(\delta+1)a}{A+a} \geq a.$$

Therefore, (11) is true.

Let $S = \limsup_{n\to\infty} x_n$, $I = \liminf_{n\to\infty} x_n$. Then (11) implies that $S \geq I \geq a > 0$, and it follows from Lemma 4.1 that

$$I \geq \frac{(\delta+1)I}{A+I} \text{ and } S \leq \frac{(\delta+1)S}{A+S},$$

which yields that $I = \delta + 1 - A = S$, i.e., $\lim_{n\to\infty} x_n = \delta + 1 - A$.

Case 2. $0 < \delta - A < 1$.

By Theorem 3.2 (b), there exists N such that $x_n > A/\delta$ for $n \geq N$. Then we have

$$x_{n+1} = \frac{\delta x_{n-m} + x_{n-k}}{A + x_{n-k}} = 1 + \frac{\delta(x_{n-m} - A/\delta)}{A + x_{n-k}} > 1$$

for $n \geq N + m$, and by Lemma 4.1 we get

$$x_{n+1} = \frac{\delta x_{n-m} + x_{n-k}}{A + x_{n-k}} \leq \frac{\delta x_{n-m} + 1}{A+1}$$

for $n \geq N + m + k + 1$. Hence

$$\limsup_{n\to\infty} x_n \leq \frac{1}{A + 1 - \delta}.$$

Let $S = \limsup_{n\to\infty} x_n$, $I = \liminf_{n\to\infty} x_n$, then

$$0 < \frac{A}{\delta} \leq I \leq S < \infty. \tag{12}$$

By Lemma 4.1, we have

$$I \geq \frac{\delta I + S}{A + S} \text{ and } S \leq \frac{\delta S + I}{A + I}.$$

This implies that

$$\delta I + S - IA \leq IS \leq \delta S + I - SA, \tag{13}$$

which yields that $(I - S)(A + 1 - \delta) \geq 0$. Hence $I = S$ since $\delta - A < 1$, and this, together with (12) and (13), leads to $I = S = \delta + 1 - A$, i.e., $\lim_{n\to\infty} x_n = \delta + 1 - A$. The proof is complete.

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