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Positive Solutions To Systems Of Nonlinear Second-Order Three-Point Boundary Value Problems*

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Abstract

In this paper, we investigate the following system of nonlinear second-order three-point boundary value problem

$$\begin{cases} -u'' = f(t,v), \ t \in (0,1), \\ -v'' = g(t,u), \ t \in (0,1), \\ u(0) = \alpha u(\eta), \ u(1) = \beta u(\eta), \\ v(0) = \alpha v(\eta), \ v(1) = \beta v(\eta), \end{cases}$$

where $\eta \in (0,1)$ and $0 < \beta \leq \alpha < 1$. Green's function for the associated linear boundary value problem is constructed, and several useful properties of the Green's function are obtained. Existence and multiplicity criteria of positive solutions are established by using the well-known fixed point theorems of cone expansion and compression.

1 Introduction

Multi-point boundary value problems (BVPs for short) of differential equations arise in a variety of applied mathematics and physics. For instance, the vibrations of a wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point BVP [8]. The study of three-point BVPs for nonlinear ordinary differential equations was initiated by Gupta [3]. Since then, nonlinear multi-point BVPs have been studied by may authors, see [1, 2, 5, 6, 7, 9, 10, 11] and the references therein. In particular, by using the Guo-Krasnosel'skii fixed point theorem in cone, Ma [6] proved the existence of at least one positive solution to the following BVP

$$\begin{cases} u''(t) + h(t)f(u) = 0, \ 0 < t < 1, \\ u(0) = 0, \ u(1) = \alpha u(\eta), \end{cases}$$

where $\eta \in (0,1)$, $0 < \alpha < \frac{1}{\eta}$, $h \in C([0,1], R^+)$ and there exists $t_0 \in [0,1]$ such that $h(t_0) > 0$, and $f \in C(R^+, R^+)$ is either superlinear or sublinear. Recently, Zhou and

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Xu [10] employed fixed point index theorems to consider the existence and multiplicity of positive solutions to the system

$$\begin{cases} -u'' = f(t,v),\ t\in(0,1),\\ -v'' = g(t,u),\ t\in(0,1),\\ u(0)=0,\ u(1)=\alpha u(\eta),\\ v(0)=0,\ v(1)=\alpha v(\eta), \end{cases}$$

where $\eta\in(0,1)$, $0<\alpha<\frac{1}{\eta}$ and $f,\ g\in C([0,1]\times R^+,\ R^+)$.

Motivated by the excellent results mentioned above, in this paper we consider the existence and multiplicity of positive solutions to the following system

$$\begin{cases} -u'' = f(t,v),\ t\in(0,1),\\ -v'' = g(t,u),\ t\in(0,1),\\ u(0)=\alpha u(\eta),\ u(1)=\beta u(\eta),\\ v(0)=\alpha v(\eta),\ v(1)=\beta v(\eta), \end{cases} \tag{1}$$

where $f,\ g\in C([0,1]\times R^+,\ R^+)$, $g(t,0)\equiv 0$, $\eta\in(0,1)$ and $0<\beta\le\alpha<1$. First, Green's function for associated linear boundary value problem is constructed, and then, several useful properties of the Green's function are obtained. Finally, some existence and multiplicity criteria of positive solutions to the system (1) are established. Our main tools are the well-known fixed point theorems of cone expansion and compression [4], which we state here for convenience of the reader.

THEOREM 1. Let E be a Banach space and P be a cone in E. Assume that Ω_1 and Ω_2 are open subsets of E such that $0\in\Omega_1,\ \overline{\Omega}_1\subset\Omega_2$, and let $T: P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ be a completely continuous operator such that either

1) $\|Tu\|\le\|u\|$ for $u\in P\cap\partial\Omega_1$ and $\|Tu\|\ge\|u\|$ for $u\in P\cap\partial\Omega_2$, or

2) $\|Tu\|\ge\|u\|$ for $u\in P\cap\partial\Omega_1$ and $\|Tu\|\le\|u\|$ for $u\in P\cap\partial\Omega_2$.

Then T has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

THEOREM 2. Let E be a Banach space and P be a cone in E. Assume that $\Omega_1,\ \Omega_2$ and Ω_3 are open subsets of E such that $0\in\Omega_1,\ \overline{\Omega}_1\subset\Omega_2,\ \overline{\Omega}_2\subset\Omega_3$, and let $T: P\cap(\overline{\Omega}_3\backslash\Omega_1)\to P$ be a completely continuous operator such that the following conditions are satisfied:

1) $\|Tu\|\ge\|u\|$ for $u\in P\cap\partial\Omega_1$;

2) $\|Tu\|\le\|u\|$ and $Tu\ne u$ for $u\in P\cap\partial\Omega_2$;

3) $\|Tu\|\ge\|u\|$ for $u\in P\cap\partial\Omega_3$.

Then T has at least two fixed points u_1 and u_2 in $P\cap(\overline{\Omega}_3\backslash\Omega_1)$. Furthermore, $u_1\in P\cap(\overline{\Omega}_2\backslash\Omega_1)$ and $u_2\in P\cap(\overline{\Omega}_3\backslash\Omega_2)$.

THEOREM 3. Let E be a Banach space and P be a cone in E. Assume that $\Omega_1,\ \Omega_2$ and Ω_3 are open subsets of E such that $0\in\Omega_1,\ \overline{\Omega}_1\subset\Omega_2,\ \overline{\Omega}_2\subset\Omega_3$, and let $T: P\cap(\overline{\Omega}_3\backslash\Omega_1)\to P$ be a completely continuous operator such that the following conditions are satisfied:

1) $\|Tu\|\le\|u\|$ for $u\in P\cap\partial\Omega_1$;

2) $\|Tu\|\ge\|u\|$ and $Tu\ne u$ for $u\in P\cap\partial\Omega_2$;

3) $\|Tu\|\le\|u\|$ for $u\in P\cap\partial\Omega_3$.

Then T has at least two fixed points u_1 and u_2 in $P\cap(\overline{\Omega}_3\backslash\Omega_1)$. Furthermore, $u_1\in P\cap(\overline{\Omega}_2\backslash\Omega_1)$ and $u_2\in P\cap(\overline{\Omega}_3\backslash\Omega_2)$.

2 Some Lemmas

Throughout this paper, we denote

$$\xi = \frac{1}{1-\alpha+(\alpha-\beta)\eta} \text{ and } \gamma = \min\left\{\frac{\beta(1-\eta)}{1-\beta\eta},\ \frac{\beta\eta}{1-\alpha+\alpha\eta}\right\}.$$

LEMMA 1. For $y \in C[0,1]$, the BVP

$$\begin{cases} -u'' = y(t),\ 0 < t < 1, \\ u(0) = \alpha u(\eta),\ u(1) = \beta u(\eta) \end{cases} \tag{2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{3}$$

where

$$G(t,s) = \xi \begin{cases} s[1-\beta\eta+(\beta-1)t], & s \le \min\{t,\eta\}, \\ t[1-\beta\eta+(\beta-1)s] + \alpha(1-\eta)(s-t), & t \le s \le \eta, \\ (1-s)(t-\alpha t+\alpha\eta) + (s-t)[1-\alpha+(\alpha-\beta)\eta], & \eta \le s \le t, \\ (1-s)(t-\alpha t+\alpha\eta), & s \ge \max\{t,\eta\} \end{cases}$$

which will be called the Green's function for the linear problem (2).

Indeed, since the unique solution of the BVP (2) can be expressed as

$$\begin{aligned} u(t) &= -\int_0^t (t-s)y(s)ds + \xi[(1-\alpha)t+\alpha\eta]\int_0^1 (1-s)y(s)ds \\ &\quad +\xi[(\alpha-\beta)t-\alpha]\int_0^\eta (\eta-s)y(s)ds, \end{aligned}$$

it is easy to verify that (3) is satisfied.

LEMMA 2. The Green's function $G(t,s)$ has the following properties:
1) $G(t,s) \ge 0$ for $0 \le t,s \le 1$
2) $G(t,s) \le G(s,s)$ for $0 \le t,s \le 1$;
3) $G(t,s) \le \xi$ for $0 \le t,s \le 1$;
4) $G(t,s) \ge \gamma G(s,s)$ for $\eta \le t \le 1$ and $0 \le s \le 1$.

Let $E = C[0,1]$ be equipped with the norm $\|u\| = \max_{0\le t\le 1}|u(t)|$ and

$$P = \{u \in E | u(t) \ge 0 \text{ for } t \in [0,1] \text{ and } \min_{\eta \le t \le 1} u(t) \ge \gamma \|u\|\}.$$

Then it is obvious that E is a Banach space and P is a cone in E. We define an operator $T : P \to E$ by

$$Tu(t) = \int_0^1 G(t,s) f\left(s, \int_0^1 G(s,r)g(r,u(r))dr\right) ds,\ t \in [0,1]. \tag{4}$$

It is easy to see that if T has a fixed point $u \in P$, then the system (1) has one solution

$$\left(u, \int_0^1 G(s,r)g(r,u(r))dr\right).$$

LEMMA 3. $T : P \to P$ is completely continuous.

PROOF. Suppose that $u \in P$. Then it follows from Lemma 2 that

$$\begin{aligned} 0 \le Tu(t) &= \int_0^1 G(t,s)f\left(s, \int_0^1 G(s,r)g(r,u(r))dr\right)ds \\ &\le \int_0^1 G(s,s)f\left(s, \int_0^1 G(s,r)g(r,u(r))dr\right)ds,\ t \in [0,1], \end{aligned}$$

which implies that $\|Tu\| \le \int_0^1 G(s,s)f\left(s, \int_0^1 G(s,r)g(r,u(r))dr\right)ds$. So,

$$\begin{aligned} \min_{\eta \le t \le 1} Tu(t) &= \min_{\eta \le t \le 1}\left\{\int_0^1 G(t,s)f\left(s, \int_0^1 G(s,r)g(r,u(r))dr\right)ds\right\} \\ &\ge \gamma \int_0^1 G(s,s)f\left(s, \int_0^1 G(s,r)g(r,u(r))dr\right)ds \ge \gamma \|Tu\|. \end{aligned}$$

Therefore, $T : P \to P$. Furthermore, we can prove by standard arguments that $T : P \to P$ is completely continuous.

3 Main Results

For a continuous function $h : [0,1] \times R^+ \to R^+$, we denote

$$h_0 = \lim_{u \to 0^+} \inf_{t \in [0,1]} \frac{h(t,u)}{u},\ h_\infty = \lim_{u \to \infty} \inf_{t \in [0,1]} \frac{h(t,u)}{u},$$

$$h^0 = \lim_{u \to 0^+} \sup_{t \in [0,1]} \frac{h(t,u)}{u},\ h^\infty = \lim_{u \to \infty} \sup_{t \in [0,1]} \frac{h(t,u)}{u}.$$

To derive the existence and multiplicity results of positive solutions to the system (1), we make the following assumptions:

(A_1) $f^0 < a$ and $g^0 < a$, where $a = \frac{1}{\int_0^1 G(r,r)dr}$;

(A_2) $f_\infty = \infty$ and $g_\infty = \infty$;

(A_3) $f_0 > b$ and $g_0 > b$, where $b = \frac{1}{\gamma \int_\eta^1 G(\frac{\eta+1}{2}, s)ds}$;

(A_4) $f^\infty = 0$ and $g^\infty = 0$;

(A_5) $f(t,u)$ and $g(t,u)$ are all nondecreasing with respect to u and there exists a constant $N > 0$ such that

$$f\left(t, \int_0^1 \xi g(s,N)ds\right) < \frac{N}{\xi},\ t \in [0,1];$$

(A_6) $f(t,u)$ is nonincreasing and $g(t,u)$ is nondecreasing with respect to u and there exists a constant $M>0$ such that

$$f\left(s,\int_0^1 G(r,r)g(r,M)dr\right)>\frac{4M}{\xi\eta\,(1-\eta)^2},\ s\in\left[\eta,\frac{1+\eta}{2}\right];$$

(A_7) $f(t,u)$ is nondecreasing and $g(t,u)$ is nonincreasing with respect to u and there exists a constant $W>0$ such that

$$f\left(s,\int_0^1 \gamma G(r,r)g(r,W)dr\right)>\frac{4W}{\xi\eta\,(1-\eta)^2},\ s\in\left[\eta,\frac{1+\eta}{2}\right].$$

THEOREM 4. Assume that (A_1) and (A_2) hold. Then the system (1) has at least one positive solution.

PROOF. By (A_1), we may choose $H_1\in(0,1)$ such that for any $(t,u)\in[0,1]\times[0,H_1)$, $f(t,u)\le au$ and $g(t,u)\le au$. Thus, if $u\in P$ and $\|u\|=\frac{H_1}{2}$, then

$$\int_0^1 G(s,r)g(r,u(r))dr\le a\int_0^1 G(r,r)u(r)dr\le\|u\|=\frac{H_1}{2}<H_1,\ s\in[0,1],$$

and so

$$\begin{aligned}Tu(t)&\le\int_0^1 G(s,s)f\left(s,\int_0^1 G(s,r)g(r,u(r))dr\right)ds\\&\le a\|u\|\int_0^1 G(s,s)a\int_0^1 G(r,r)drds=\|u\|,\ t\in[0,1].\end{aligned}$$

If we let $\Omega_1=\{u\in E:\|u\|<\frac{H_1}{2}\}$, then we have shown that

$$\|Tu\|\le\|u\|\ \text{ for } u\in P\cap\partial\Omega_1. \tag{5}$$

On the other hand, in view of (A_2), there exist two positive numbers C_1 and C_2 such that for any $(t,u)\in[0,1]\times R^+$, $f(t,u)\ge\varphi u-C_1$ and $g(t,u)\ge\psi u-C_2$, where $\varphi>0$ and $\psi>0$ satisfy $\varphi\int_\eta^1 G\left(\frac{\eta+1}{2},s\right)ds\ge 2$ and $\psi\gamma^2\int_\eta^1 G(s,s)ds\ge 1$. Then for $u\in P$, we have

$$\begin{aligned}Tu\left(\frac{\eta+1}{2}\right)&=\int_0^1 G\left(\frac{\eta+1}{2},s\right)f\left(s,\int_0^1 G(s,r)g(r,u(r))dr\right)ds\\&\ge\int_0^1 G\left(\frac{\eta+1}{2},s\right)\left[\varphi\int_0^1 G(s,r)g(r,u(r)dr-C_1\right]ds\\&\ge 2\|u\|-C_3,\end{aligned}$$

where $C_3=\varphi C_2\int_0^1 G\left(\frac{\eta+1}{2},s\right)\int_0^1 G(r,r)drds+C_1\int_0^1 G\left(\frac{\eta+1}{2},s\right)ds$. Consequently, $\|Tu\|\ge Tu(\frac{\eta+1}{2})\ge\|u\|$ as $\|u\|\to\infty$. Thus, for $H_2>0$ large enough, if we let $\Omega_2=\{u\in E:\|u\|<H_2\}$, then we have

$$\|Tu\|\ge\|u\|\ \text{ for } u\in P\cap\partial\Omega_2. \tag{6}$$

Therefore, it follows from (5), (6) and Theorem 1 that T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$, which implies that the system (1) has at least one positive solution.

THEOREM 5. Assume that (A_3) and (A_4) hold. Then the system (1) has at least one positive solution.

PROOF. By (A_3), there exists $\widehat{H}_1 \in (0,1)$ such that for any $(t,u) \in [0,1] \times [0, \widehat{H}_1]$, $f(t,u) \geq bu$ and $g(t,u) \geq bu$. In view of $g(t,0) \equiv 0$ and the continuity of $g(t,u)$, we can choose $H_3 \in (0, \widehat{H}_1)$ sufficiently small such that

$$g(t,u) \leq \frac{\widehat{H}_1}{\int_0^1 G(r,r)dr}, \ (t,u) \in [0,1] \times [0, H_3].$$

Thus, if $u \in P$ and $\|u\| = H_3$, then

$$\int_0^1 G(s,r)g(r,u(r))dr \leq \frac{\widehat{H}_1}{\int_0^1 G(r,r)dr} \int_0^1 G(s,r)dr \leq \widehat{H}_1, \ s \in [0,1],$$

and so

$$\begin{aligned} Tu\left(\frac{\eta+1}{2}\right) &= \int_0^1 G\left(\frac{\eta+1}{2}, s\right) f\left(s, \int_0^1 G(s,r)g(r,u(r))dr\right) ds \\ &\geq b^2 \int_\eta^1 G\left(\frac{\eta+1}{2}, s\right) \int_\eta^1 G(s,r)u(r)drds \\ &\geq b^2\gamma^2 \|u\| \int_\eta^1 G(r,r)dr \int_\eta^1 G(\frac{\eta+1}{2}, s)ds \\ &\geq \|u\|. \end{aligned}$$

If we let $\Omega_3 = \{u \in E : \|u\| < H_3\}$, then we have shown that

$$\|Tu\| \geq \|u\| \ \text{ for } u \in P \cap \partial\Omega_3. \tag{7}$$

On the other hand, in view of (A_4), there exist two positive numbers C_4 and C_5 such that for any $(t,u) \in [0,1] \times R^+$, $f(t,u) \leq \lambda u + C_4$ and $g(t,u) \leq \lambda u + C_5$, where $\lambda > 0$ satisfies $\lambda \int_0^1 G(r,r)dr \leq \frac{1}{2}$. Then for $u \in P$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f\left(s, \int_0^1 G(s,r)g(r,u(r))dr\right) ds \\ &\leq \int_0^1 G(s,s)\left[\lambda \int_0^1 G(r,r)g(r,u(r))dr + C_4\right] ds \\ &\leq \lambda \int_0^1 G(s,s)ds \int_0^1 G(r,r)\left[\lambda u(r) + C_5\right] dr + C_4 \int_0^1 G(s,s)ds \\ &\leq \frac{1}{4}\|u\| + C_6, \ t \in [0,1], \end{aligned}$$

where $C_6 = \left(\lambda C_5 \int_0^1 G(r,r)dr + C_4\right) \int_0^1 G(s,s)ds$. Consequently, $\|Tu\| \leq \|u\|$ as $\|u\| \to \infty$. Thus, for $H_4 > 0$ large enough, if we let $\Omega_4 = \{u \in E : \|u\| < H_4\}$, then we have

$$\|Tu\| \leq \|u\| \ \text{ for } u \in P \cap \partial\Omega_4. \tag{8}$$

Therefore, it follows from (7), (8) and Theorem 1 that T has a fixed point in $P \cap (\overline{\Omega}_4 \backslash \Omega_3)$, which implies that the system (1) has at least one positive solution.

THEOREM 6. Assume that (A_2), (A_3) and (A_5) hold. Then the system (1) has at least two positive solutions.

PROOF. Let $B_N = \{u \in E : \|u\| < N\}$. Then by Lemma 2 and (A_5), for any $u \in P \cap \partial B_N$, we have

$$Tu(t) \leq \xi \int_0^1 f\left(s, \int_0^1 \xi g(r, N)dr\right) ds < \xi \frac{N}{\xi} = N, \ t \in [0, 1],$$

which shows that

$$\|Tu\| < \|u\| \ \text{ for } u \in P \cap \partial B_N. \tag{9}$$

In view of (A_2), (A_3) and the proof of Theorem 4 and Theorem 5, we can choose $H_3 < N < H_2$ such that

$$\|Tu\| \geq \|u\| \ \text{ for } u \in P \cap \partial\Omega_2 \tag{10}$$

and

$$\|Tu\| \geq \|u\| \ \text{ for } u \in P \cap \partial\Omega_3. \tag{11}$$

Therefore, it follows from (9)-(11) and Theorem 2 that T has a fixed point in $P \cap (\overline{\Omega}_2 \backslash B_N)$ and a fixed point in $P \cap (\overline{B}_N \backslash \Omega_3)$, which implies that the system (1) has at least two positive solutions.

THEOREM 7. Assume that (A_1), (A_4) and (A_6) hold. Then the system (1) has at least two positive solutions.

PROOF. Let $B_M = \{u \in E : \|u\| < M\}$. Then by Lemma 2 and (A_6), for any $u \in P \cap \partial B_M$, we have

$$\begin{aligned} Tu(\eta) &= \int_0^1 G(\eta, s) f\left(s, \int_0^1 G(s, r) g(r, u(r)) dr\right) ds \\ &\geq \int_\eta^{\frac{1+\eta}{2}} G(\eta, s) f\left(s, \int_0^1 G(r, r) g(r, M) dr\right) ds \\ &\geq \frac{\xi\eta\,(1-\eta)}{2} \int_\eta^{\frac{1+\eta}{2}} f\left(s, \int_0^1 G(r, r) g(r, M) dr\right) ds \\ &> \frac{\xi\eta\,(1-\eta)}{2} \int_\eta^{\frac{1+\eta}{2}} \frac{4M}{\xi\eta\,(1-\eta)^2} ds = M, \end{aligned}$$

which shows that

$$\|Tu\| > \|u\| \ \text{ for } u \in P \cap \partial B_M. \tag{12}$$

In view of (A_1), (A_4) and the proof of Theorem 4 and Theorem 5, we can choose $H_1 < M < H_4$ such that

$$\|Tu\| \leq \|u\| \ \text{ for } u \in P \cap \partial\Omega_1 \tag{13}$$

and

$$\|Tu\| \le \|u\| \text{ for } u \in P \cap \partial\Omega_4. \tag{14}$$

Therefore, it follows from (12)-(14) and Theorem 3 that T has a fixed point in $P \cap (\overline{\Omega}_4 \backslash B_M)$ and a fixed point in $P \cap (\overline{B}_M \backslash \Omega_1)$, which implies that the system (1) has at least two positive solutions.

THEOREM 8. Assume that (A_1), (A_4) and (A_7) hold. Then the system (1) has at least two positive solutions.

Since the proof of this theorem is similar to that of Theorem 7, we have omitted it.

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