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Common Fixed Points Versus Invariant Approximation In Nonconvex Sets*

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Abstract

The aim of the present paper is to establish an existence result on common fixed point of best approximation without using the starshapedness condition of the domain. As a consequence, our result improves and extends the corresponding results of Dhage [4] and Mukherjee and Som [10].

1 Introduction

Existence of fixed point has been used at many places in approximation theory. Several results exist in the literature where fixed point theorems are used to prove the existence of best approximation. Meinardus [9] was the first who employed a fixed point theorem to establish the existence of an invariant approximation. Afterwards, Brosowski [1] obtained a celebrated result and generalized the Meinardus's result. Using another fixed-point theorem, Subrahmanyam [14] obtained generalization of the result of Meinardus [9]. Further, Singh [12] observed that the linearity of mapping and the convexity of the set of best approximations in the result of Brosowski [1], can be relaxed. In a subsequent paper, Singh [13] also observed that only the nonexpansiveness of mapping on the set of best approximations is necessary for the validity of his own earlier result in [12]. Later, Hicks and Humpheries [7] showed that the result of Singh [12] remains true, if domain of mapping is replaced by the boundary of domain. Furthermore, Sahab et al. [11] generalized the result of Hicks and Humpheries [7] and Singh [12] by using a pair of commuting mappings, one linear and the other nonexpansive and established the existence of best approximation in a normed space.

Recently, Dhage [4] obtained an existence result similar to Theorem 3 of Sahab et al. [11] concerning the existence of a best approximation from a subset to two points of a normed linear space under weaker conditions.

Dotson [5] proved the existence of a fixed point for a nonexpansive mapping. He further extended his result beyond the star-shaped domain in [6]. This idea was

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utilized by Mukherjee and Som [10] to prove existence of fixed point and then to apply it for proving existence of invariant approximation. In this sense, they extended the result of Singh [12] without the starshapedness condition.

The purpose of this paper is to show the validity of Theorem 3.2 of Dhage [4] without using the starshapedness condition of domain in a normed linear space. As a by-product, the result of Mukherjee and Som [10] is also extended.

2 Preliminaries

We recall some definitions.

DEFINITION 2.1. [4]. Let X be a normed linear space with norm $\|.\|$ and let C be a nonempty subset of X. Let $x_0 \in X$. An element $y \in C$ is called a best approximant to $x_0 \in X$, if

$$\|x_0 - y\| = d(x_0, C) = \inf\{\|x_0 - z\| : z \in C\}.$$

Let $\mathcal{A}_{\mathcal{C}}(x_0)$ be the set of best C-approximants to x_0 and so

$$\mathcal{A}_{\mathcal{C}}(x_0) = \{y \in C : \|x_0 - y\| = d(x_0, C)\}. \tag{1}$$

Denote

$$\mathcal{A}'_{\mathcal{C}}(x_0) = \mathcal{A}_{\mathcal{C}}(x_0) \cup \{x_0\}. \tag{2}$$

DEFINITION 2.2. [4]. (i) A subset C of X is said to be convex, if $\lambda x + (1-\lambda)y \in C$, whenever $x, y \in C$ and $0 \leq \lambda \leq 1$. (ii) A subset C of X is said to be starshaped, if there exists at least one point $p \in C$ such that the line segment joining x to p is contained in C for all $x \in C$, that is $\lambda x + (1-\lambda)p \in C$, for all $x \in C$ and $0 < \lambda < 1$. In this case p is called the star center of C.

Each convex set is starshaped with respect to each of its points, but not conversely.

DEFINITION 2.3. A pair of self-mappings (I, T) of a normed linear space X is said to be commutative on C, if $ITx = TIx$ for all $x \in C$.

DEFINITION 2.4. [2]. A non-empty set X with a function $\rho : X \times X \times X \to (0, \infty)$ is called a D-metric space with a D-metric ρ, denoted by (X, ρ) if ρ satisfies:

(i) $\rho(x, y, z) = 0 \Leftrightarrow x = y = z$ (coincidence),

(ii) $\rho(x, y, z) = \rho(p\{x, y, z\})$ where p is a permutation of $\{x, y, z\}$ (symmetry), and

(iii) $\rho(x, y, z) \leq \rho(x, y, a) + \rho(x, a, z) + \rho(a, y, z)$ for all $x, y, z, a \in X$ (tetrahedral inequality).

For details we refer to Dhage [4].

DEFINITION 2.5. [4]. (i) A sequence $\{x_n\} \subset X$ is called convergent to $x \in X$ if $\lim_{m,n} \ \rho(x_m, x_n, x) = 0$. (ii) A sequence $\{x_n\} \subset X$ is called D-Cauchy if $\lim_{m,n,p} \ \rho(x_m, x_n, x_p) = 0$. (iii) A complete D-metric space is the one in which every D-Cauchy sequence converges to a point in it.

It has been shown in [2] and [8, Lemma 2.9] that the D-metric ρ is a continuous function on $X \times X \times X$ in the topology of D-metric convergence which is Hausdorff.

DEFINITION 2.5. [4]. A mapping $f : (X, \rho) \to (X, \rho)$ is said to D-contractive with respect to a mapping $g : (X, \rho) \to (X, \rho)$, if

$$\rho(fx, fy, fz) < \rho(gx, gy, gz) \tag{3}$$

for all $x, y, z \in X$ for which $\rho(gx, gy, gz) \neq 0$, and D-nonexpansive with respect to g, if

$$\rho(fx, fy, fz) \leq \rho(gx, gy, gz) \tag{4}$$

for all $x, y, z \in X$.

DEFINITION 2.7. [4]. Two maps $f, g : (X, \rho) \to (X, \rho)$ are called coincident if there is an $x \in X$ such that $fx = gx$ and limit coincident if there is a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n$. Similarly they are called limit commutative or limit commuting if there is a sequence $\{x_n\}$ in X such that $\lim_n(fg)(x_n) = \lim_n(gf)(x_n)$. Two maps $f, g : (X, \rho) \to (X, \rho)$ are called limit coincidentally commuting if their limit coincidence implies the limit commutingness on X, i.e., there exists a sequence $\{x_n\}$ in X such that $\lim_n fx_n = \lim_n gx_n$ implies $\lim_n(fg)(x_n) = \lim_n(gf)(x_n)$. Similarly, two maps $f, g : (X, \rho) \to (X, \rho)$ are called coincidentally commuting if they commute at coincidence points. Finally, a mapping $f : (X, \rho) \to (X, \rho)$ is continuous if and only if for any sequence $\{x_n\}$ in X, $x_n \to x$ implies $fx_n \to fx$.

We remark that every commuting pair of maps on a D-metric space is limit coincidentally commuting, but the converse may not be true. Similarly limit coincidentally commutativity implies coincidentally commutativity, but the converse may not be true (for detail, see [2]).

DEFINITION 2.8. [2]. For $x_0, y_0 \in X$, let us denote

$$D(x_0, y_0, C) = \inf\{\rho(x_0, y_0, c) : c \in C\}. \tag{5}$$

An element $z \in C$ is said to be a best approximant to x_0 and y_0 from C (or closest to x_0 and y_0 from C) if

$$\rho(x_0, y_0, z) = D(x_0, y_0, C).$$

In this case the element z is called a best C-approximant to x_0 and y_0 from C and the set of all such best C-approximants to x_0 and y_0 from C is denoted by

$$\mathcal{A}_{\mathcal{C}}(x_0, y_0) = \{z \in C : \rho(x_0, y_0, z) = D(x_0, y_0, C)\}. \tag{6}$$

Denote

$$\mathcal{A}'_{\mathcal{C}}(x_0, y_0) = \mathcal{A}_{\mathcal{C}}(x_0, y_0) \cup \{x_0, y_0\}. \tag{7}$$

It is remarked that the notion of a closest element to x_0 and y_0 from a subset C is different from that of element to the set $\{x_0, y_0\}$ from C (cf.[4]).

We give the definition of contractive jointly continuous family introduced by Dotson [6].

DEFINITION 2.9. [6]. Let $F = \{f_\alpha\}_{\alpha \in X}$ be a family of functions from $[0, 1]$ into a normed linear space X such that $f_\alpha(1) = \alpha$ for each $\alpha \in X$. The family F is said to be contractive, if there exists a function $\phi : (0,1) \to (0,1)$ such that for all $\alpha, \beta \in X$ and all $t \in (0,1)$,we have

$$\|f_\alpha(t) - f_\beta(t)\| \le \phi(t)\|\alpha - \beta\|.$$

The family F is said to be jointly continuous(resp.jointly weakly continuous) if $t \to t_0$ in $[0,1]$ and $\alpha \to \alpha_0$ in X(resp. if $t \to t_0$ in $[0,1]$ and $\alpha \to^w \alpha_0$ in X), then $f_\alpha(t) \to f_{\alpha_0}(t_0)$ (resp. $f_\alpha(t) \to^w f_{\alpha_0}(t_0)$) in X; here $\to$ and $\to^w$ denote the strong and weak convergence, respectively.

REMARK 2.10. In the light of the comments by Dotson [6], if $C \subseteq X$ is $p-$starshaped and $f_\alpha(t) = (1-t)p + t\alpha$, $(\alpha \in C, t \in [0,1])$, then $\{f_\alpha\}_{\alpha \in C}$ is a contractive jointly continuous family with $\phi(t) = t$. Thus the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which is turn contains the class of convex sets.

The following result is needed in the sequel.

THEOREM 2.11. [3] Let T and g be two continuous selfmappings of a compact D-metric space X satisfying (3). Further, suppose that $T(X) \subseteq g(X)$ and $\{T, g\}$ are coincidentally commuting. Then T and g have a unique common fixed point.

3 Main Result

To prove our result, we use the technique of Dhage [4].

THEOREM 3.1. Let C be a nonempty subset of a normed linear space X and $\{x_0, y_0\} \subset X$. Let $T, g : X \to X$ be two mappings satisfying the following conditions:
(i) T is nonexpansive on $\Delta = \mathcal{A}_c(x_0, y_0)$ w.r.t. the map g,
(ii) $T : \partial C \to C$, where ∂C is a boundary of C,
(iii) $\{T, g\} : \{x_0, y_0\} \to \{x_0, y_0\}$ are injective,
(iv) $\{T, g\}$ are limit coincidentally commuting on Δ and
(v) g is uniformly continuous on X.
Further, if Δ is nonempty compact and has a contractive jointly continuous family $F = \{f_\alpha\}_{\alpha \in \Delta}$, and $g(\Delta) \subseteq \Delta$, then T and g have a common fixed point in C which is closet to x_0 and y_0 w.r.t. a D-metric ρ on X defined by $\rho(x,y,z) = \|x-y\| + \|y-z\| + \|z-x\|$.

PROOF. Define D-metric ρ on the normed linear space X as:

$$\rho(x,y,z) = \|x-y\| + \|y-z\| + \|z-x\| \tag{8}$$

for $x, y, z \in X$. Since T is nonexpansive on X w.r.t. map g, we have

$$\begin{aligned} \rho(Tx, Ty, Tz) &= \|Tx - Ty\| + \|Ty - Tz\| + \|Tz - Tx\| \\ &\le \|gx - gy\| + \|gy - gz\| + \|gz - gx\| \\ &= \rho(gx, gy, gz) \end{aligned}$$

for all $x, y, z \in X$ which shows that T is also D-nonexpansive on X w.r.t. the map g.

Let $y \in \Delta$. Then $gy \in \Delta$, because $g(\Delta) \subseteq \Delta$. It is remarked by Hicks and Humpheries [7] that the element of best C-approximation need not belong to the interior of C, so $y \in \partial C$. Since $T(\partial C) \subset C$, we have $Ty \in C$ and from (6), it follows that

$$\begin{aligned} \rho(Ty, x_0, y_0) &= \rho(Ty, Tx_0, Ty_0) \\ &\leq \rho(gy, gx_0, gy_0) \\ &= \rho(gy, x_0, y_0) \\ &= D(x_0, y_0, C) \end{aligned}$$

and, therefore, $Ty \in \Delta$.

Choose a sequence $k_n \in (0,1)$ with $\{k_n\} \to 1$ as $n \to \infty$. Define a sequence $\{T_m\}$ of mappings on Δ by

$$T_m x = f_{Tx}(k_m)$$

for each $m \in \mathbb{N}$. From hypothesis (i) it follows that T and consequently each T_m is uniformly continuous on Δ. From the definition of contractive of F, each T_m is well defined from Δ into Δ for each $m \in \mathbb{N}$. Further, we show that $\{T_m, g\}$ satisfy the condition (3) and are limit coincidentally commuting on Δ for sufficiently large values of m.

First, we show that $\{T_m, g\}$ satisfy condition (3) on Δ for each $m \in \mathbb{N}$. Since T is nonexpansive on Δ w.r.t. the map g, we have

$$\begin{aligned} \rho(T_m x, T_m y, T_m z) &= \|T_m x - T_m y\| + \|T_m y - T_m z\| + \|T_m z - T_m x\| \\ &= \|f_{Tx}(k_m) - f_{Ty}(k_m)\| + \|f_{Ty}(k_m) - f_{Tz}(k_m)\| \\ &\quad + \|f_{Tz}(k_m) - f_{Tx}(k_m)\| \\ &\leq \phi(k_m)\|Tx - Ty\| + \phi(k_m)\|Ty - Tz\| + \phi(k_m)\|Tz - Tx\| \\ &\leq \phi(k_m)\|gx - gy\| + \phi(k_m)\|gy - gz\| + \phi(k_m)\|gz - gx\| \\ &\leq \phi(k_m)[\|gx - gy\| + \|gy - gz\| + \|gz - gx\|] \\ &= \phi(k_m)\rho(gx, gy, gz) \\ &< \rho(gx, gy, gz) \ \ (\phi(k_m) < 1) \end{aligned}$$

for all $x, y, z \in \Delta$ for which $\rho(gx, gy, gz) \neq \phi$ (since $\rho \in (0, \infty)$). Thus $\{T_m, g\}$ satisfy condition (3) on Δ and hence are D-contractive on Δ.

Secondly, we show that $\{T_m, g\}$ are limit coincidentally commuting on Δ for sufficiently large values of m. Assume that $\{T_m, g\}$ are limit coincident on Δ for large values of m, that is, there is a sequence $\{x_n\}$ in Δ such that

$$\lim_n T_m x_n = \lim_n g x_n \ \text{ for large values of } \ m.$$

By the definition of T_m,

$$\begin{aligned} \lim_n(\lim_m T_m x_n) &= \lim_n(\lim_m f_{Tx_n}(k_m)) \\ &= \lim_n f_{Tx_n}(1) \ \ (\text{by } \ k_m \to 1 \ \text{ and jointly continuity of } \Delta) \\ &= \lim_n Tx_n \\ &= \lim_n gx_n \ \ (\text{by limit coincidentally commutingness of } T \text{ and } g) \end{aligned}$$

i.e.

$$\lim_n(\lim_m T_m x_n) = \lim_n gx_n.$$

Now,

$$\begin{aligned}\lim_m T_m(gx_n) &= \lim_m f_{Tgx_n}(k_m)\\ &= f_{Tgx_n}(1)\ \text{(by } k_m \to 1 \text{ and jointly continuouity of } \Delta)\\ &= Tgx_n.\end{aligned}$$

Therefore, by the uniform continuity of g,

$$\begin{aligned}\lim_n(\lim_m T_m(gx_n)) &= \lim_n Tgx_n\\ &= \lim_n gTx_n\\ &= g\lim_n Tx_n\\ &= g(\lim_n(\lim_m T_m x_n))\\ &= \lim_n(g\lim_m T_m x_n)\\ &= \lim_n(\lim_m gT_m x_n)\end{aligned}$$

or,

$$\lim_n(\lim_m T_m(gx_n)) = \lim_n(\lim_m g(T_m x_n))$$

which shows that T_m and g are limit coincidentally commuting on Δ for sufficiently large values of m.

Since the norm $\|.\|$ and the D-metric ρ defined by (8) generate equivalent topologies on X, therefore, the compactness of Δ and the continuity of g w.r.t. the norm $\|.\|$ imply the compactness of Δ and the continuity of g w.r.t. the D-metric ρ on X [2]. Now, Theorem 2.11 guarantees that T_m and g have a unique common fixed point x_n in Δ for sufficiently large values of m, i.e., we have

$$x_n = T_m x_n = gx_n$$

for sufficiently large values of m.

The compactness of Δ implies that the sequence $\{x_n\}$ has a convergent subsequence, say $\{x_{m_i}\}$ converging to a point $z \in \Delta$.

By the definition of T_{m_i}

$$x_{m_i} = T_{m_i} x_{m_i} = f_{Tx_{m_i}}(k_{m_i}). \tag{9}$$

Since g is continuous, the D-nonexpansiveness of T w.r.t. g implies that T is also continuous on Δ w.r.t. the D-metric ρ on it. Therefore, we have from joint continuity of F, as $i \to \infty$ in (9)

$$z = \lim_i x_{m_i} = \lim_i T_{m_i} x_{m_i} = \lim_i f_{Tx_{m_i}}(k_{m_i}) = Tz.$$

Similarly,

$$gz = g(\lim_i x_{m_i}) = \lim_i gx_{m_i} = \lim_i x_{m_i} = z.$$

Hence z is a common fixed point of T and g in C which is closest to x_0 and y_0. This completes the proof.

REMARK 3.2. Theorem 3.1 also remains true if we replace the D-metric ρ given in (8) by

$$\rho(x, y, z) = \max\{\|x - y\|, \|y - z\|, \|y - z\|\}. \tag{10}$$

REMARK 3.3. In the light of Remark 2.10, our Theorem 3.1 generalizes Theorem 3.2 due to Dhage [4].

REMARK 3.4. When $x_0 = y_0$ and $\rho(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, where d is an ordinary metric on the normed linear space X defined by $d(x, y) = \|x - y\|$, our Theorem 3.1 generalizes Theorem 2 of Mukherjee and Som [10] for a pair of maps.

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